Algebraic Topology   **Assessed Exercises II**

To be handed in by 12.00 Friday of Week 2 of Term 3

1. At an abstract level, we can form a 1-point compactification of $\mathbb{C}$ by adding an extra point ("∞") and defining the topology on $\mathbb{C}\cup\{\infty\}$ to have, as its open sets, all subsets of $\mathbb{C}$ open in the usual (Euclidean) topology on $\mathbb{C}$, together with those sets set containing the point ∞ whose complement, which is of course a subset of $\mathbb{C}$, is closed and bounded in the usual topology.

a) Show that the usual stereographic projection $S^2 \smallsetminus \{\text{North Pole}\} \to \mathbb{C}$ extends to a homeomorphism from S^2 to the space $\mathbb{C}\cup\{\infty\}$ just described.

b) Show that a polynomial map $f : \mathbb{C} \to \mathbb{C}$ can be extended (in a unique way) to a continuous map $\hat{f} : S^2 \to S^2$, where $S^2 = \mathbb{C}\cup\{\infty\}$.

c) Show that if $f \in \mathbb{C}[z]$, $z_0 \in \mathbb{C}$ and $f(z_0) = a$, then the local degree of f at z_0 (as in Hatcher, page 136) is equal to the multiplicity of z_0 as a root of $f-a$. Hint: write $f(z)-a = (z-z_0)^k g(z)$ with $g(z_0) \neq 0$. Show that on a sufficiently small circle S^1_r centred at z_0, the map

$$S^1_r \to S^1, \quad z \mapsto \frac{f(z)-a}{|f(z)-a|}$$

is homotopic to the map

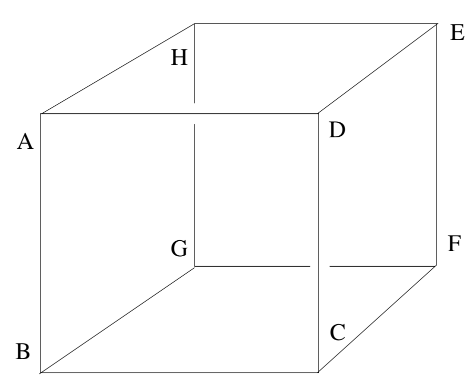
$$z \mapsto \frac{(z-z_0)^k}{|(z-z_0)^k|}.$$

d) Show that the local degree of $\hat{f}$ at ∞ is equal to the (polynomial) degree of f.

e) Prove the fundamental theorem of algebra: every polynomial $f \in \mathbb{C}[z]$ has a (complex) root.

2. In Example 2.39, right-hand diagram, Hatcher's book describes a space obtained as a quotient of the 3-dimensional unit cube by changing the equivalence relation which gives rise to the 3-torus. Write a clear account of the calculation of its homology groups, filling in details missing from Hatcher's account. Make clear also that, as Hatcher says, this space is homeomorphic to the Cartesian product of a Klein bottle and a circle.

3. Let X be the space obtained as a quotient of the 3-dimensional cube by identifying each 2-dimensional face with the face opposite by means of a clockwise twist through $\pi/2$ followed by a translation. Thus the front face is identified with the back by identifying A with E, B with H, C with G and D with F, the left face is identified with the right by identifying D with H, E with G, F with B, and C with A, and similarly for the top and bottom faces.



a) Show that X has a CW structure with one 3-cell, three 2-cells, four 1-cells and two 0-cells.

b) Compute the homology groups of X using cellular homology. You do not need to give as much detail here as in Exercise 2.

c) Let f : Cube $\to$ Cube be clockwise rotation through $2\pi/3$ about the axis DG, sending A, E, C to E, C, A and B, H, F to H, F, B, respectively. Show that f passes to the quotient to define a map $X \to X$, and compute $f_* : H_i(X) \to H_i(X)$ for $i = 0, 1, 2, 3$.

4. The surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between their boundary surfaces M_g, form a compact 3-manifold (without boundary), X. Compute the homology groups of X by using the Mayer-Vietoris sequence for this decomposition of X into two copies of R – explaining first why Mayer-Vietoris can be used here. Also compute the relative homology groups $H_i(R, M_g)$.