8 Lecture 8

Let G be a group acting on a topological space X.

Definition 8.1. Define $\text{Fix}(G) = \bigcap_{g \in G} \text{Fix}(g)$. If $\text{Fix}(G) \neq \emptyset$, then we say G has a *global fixed point*, that is an $x \in X$ such that $g \cdot X = x$ for all $g \in G$.

Exercise 8.2. Classify discrete $G < \text{Isom}(\mathbb{E}^2)$ such that G has a local fixed point. Hint: $G \cong C_n = \langle \alpha \mid \alpha^n \rangle$ or $G \cong D_{2n} = \langle \alpha, \beta \mid \alpha^2, \beta^2, (\alpha\beta)^2 \rangle \cong \langle \alpha, \gamma \mid \alpha^2, \gamma^n, (\alpha\gamma)^2 \rangle$. Recall, for example, that D_8 is the symmetry group of a square:

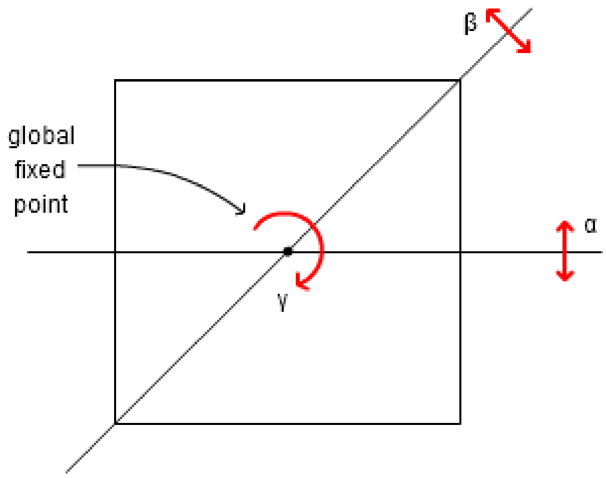

Figure 1: Here α and β are reflections generating D_8.

Definition 8.3. Let X be a geometry. Say $G < \text{Isom}(X)$ has an *invariant line* if there is a geodesic $L \subset X$ such that $g(L) = L$ for all $g \in G$.

Example 8.4. If $\alpha \in \text{Isom}(\mathbb{E}^2)$ is a rotation by π about a point $P \in L \subset \mathbb{E}^2$, then $\langle \alpha \rangle$ has L as an invariant line. If $\beta \in \text{Isom}(\mathbb{E}^2)$ is a reflection, then $\langle \beta \rangle$ leaves L invariant.

Exercise 8.5. Up to rescaling and conjugacy, every discrete group $G < \text{Isom}(\mathbb{E}^2)$ which (i) has an invariant line, and (ii) has $\text{Fix}(G) = \emptyset$ was mentioned last lecture: these are the *frieze* groups. That is, $G < \text{Isom}(\mathbb{E}^2)$ discrete with no global fixed point has an invariant line if and only if it is a frieze group. Classify these.

Why is discreteness a necessary hypothesis? If we allow indiscrete groups, there will be "too many" to classify! That is, discreteness corresponds to having "nice" combinatorial, topological and geometrical properties.

Remark 8.6. Even if we restrict to discrete groups, up to conjugacy, we many still have uncountably many. But these typically form nice *moduli spaces* (i.e., hopefully a manifold!). For example, if $a > 0$ and $G_a = \langle T_a \rangle < \text{Isom}(\mathbb{E}^2)$ then $\mathbb{E}^2/G_a \cong \mathbb{S}^1_{(a)} \times \mathbb{R}$, an annulus of circumference a.

Exercise 8.7. If $G < \text{Isom}(X)$ is discrete and acts freely, then $F = X/G$ is a manifold with a geometry modelled on X.

Definition 8.8. Let X be a geometry and $G <$ X discrete. Then G acting on X is called *cocompact* if X/G is compact.

Example 8.9. Any such $G < \text{Isom}(\mathbb{S}^n)$ is cocompact because $\mathbb{S}^n$ is compact.

Remark 8.10. The frieze groups are not cocompact.

Definition 8.11. Call $G < \text{Isom}(\mathbb{E}^2)$ a *wallpaper group* if G is discrete and acts cocompactly.

Our challenge is to classify the wallpaper groups into families, up to conjugacy and scaling.

Question 8.12. Why the mania with "classification"?

Answer 8.13. Because we are mathematicians! Classifying objects ensures we know all examples, and can deal with them accordingly. Classification can also be used to solve problems later, for example when using "proof by cases".

Example 8.14. Pick any three non-collinear points $P, Q, R \in \mathbb{E}^2$, and let α, β, γ be rotations by π about P, Q, R respectively. Let $G = \langle \alpha, \beta, \gamma \rangle < \text{Isom}(\mathbb{E}^2)$. We generate the pattern of lines shown in Figure 2.

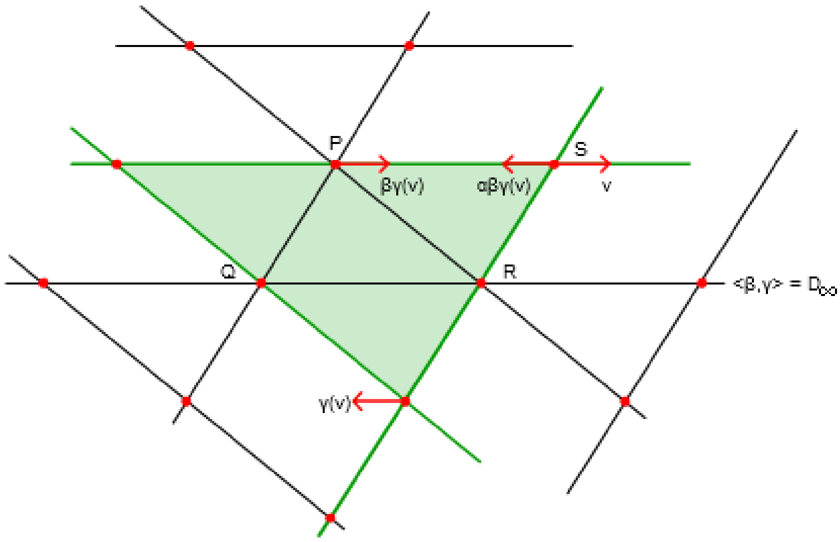

Figure 2: A wallpaper group

Each black line corresponds to a subgroup of G isomorphic to D_∞. In addition $\alpha\beta\gamma(v) = -v$, so $\delta = \alpha\beta\gamma$ is again a rotation about the point $S \in \mathbb{E}^2$ by π.

Furthermore:

Exercise 8.15. (i) G is discrete, $G \cong \langle \alpha, \beta, \gamma, \delta \mid \alpha^2, \beta^2, \gamma^2, \delta^2, \alpha\beta\gamma\delta \rangle \cong \langle \alpha, \beta, \gamma \mid \alpha^2, \beta^2, \gamma^2, (\alpha\beta\gamma)^2 \rangle$.

(ii) To prove this, show that the green triangle in the diagram is a fundamental domain for G.

(iii) The quotient $\mathbb{E}^2/G$ is $\mathbb{S}^2(2, 2, 2, 2)$, the 2-sphere with four cone points of order 2 (see diagram 3).

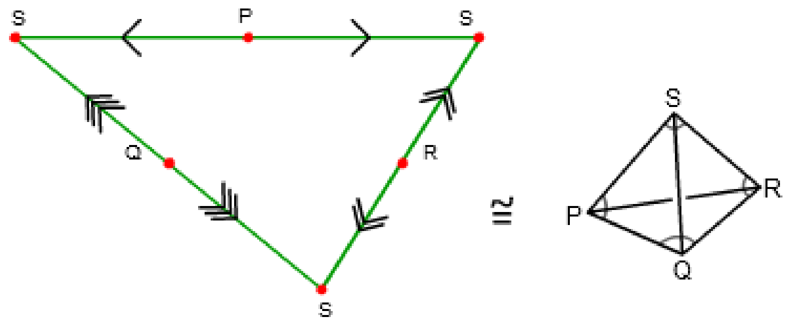

Figure 3: The green triangle with identifications

(The corners of the tetrahedron are the cone points.)