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Sam Hatfield 20th Jan 2012

6 Lecture 6

Recall: $\text{Isom}(\mathbb{E}^2) \cong \mathbb{R}^2 \rtimes \text{O}(2) \cong \text{O}(2) \ltimes \mathbb{R}^2$.
Elements of $\mathbb{R}^2$ are translations and elements of O(2) are rotations about the origin. Recall that the symbol $\ltimes$ represents a semi-direct product. If $G = A \ltimes B$ then B is a normal subgroup of G.

Example 6.1. In dimension one: We have $\text{Isom}(\mathbb{E}^1) = \mathbb{R}^2 \rtimes \text{O}(1)$ where $\text{O}(1) \cong < R_0 > \cong \mathbb{Z}_2$.We now use the classification of elements in $\text{Isom}(\mathbb{E}^1)$ to show that any element is a product of reflections. There are three kinds of elements: the identity, reflections in a point, and rotations.
Reflection: obvious
Translation: If a translation by a is T_a then we have $T_a = R_{\frac{a}{2}} \circ R_0 = R_{\frac{a}{2}+b} \circ R_b$, therefore T can be expressed in many numbers of ways by varying b.

FIGURE

So $R_d \circ R_a = T_{2(d-c)}$ the product of two reflections is translation by twice their seperation.

In dimension two: Now we use the classification of elements of $\text{Isom}(\mathbb{E}^2)$ to show any element is the product of reflections:

- The identity is the product of zero reflections.
- One reflection is the product of one reflection.
- Now for two reflections. If α and β are reflections about the lines M and N respectively then there are two cases:

  FIGURE

  (i) If $M \cap N = \{x\}$ then $\beta \circ \alpha$ is rotation around x by twice the angle between them.

  (ii) If m and N are parallel ($M \cap N = \varnothing$) then $\beta \circ \alpha$ is translation by twice $\text{d}_{\mathbb{E}}(M, N)$ in direction perpendicular to both, from M towards N.

  FIGURE
- A glide reflection is the product of three reflections. $\gamma \circ \beta \circ \alpha = \beta \circ \alpha \circ \gamma$ is a glide reflection along P.

Note: Translations and glide reflections act freely but rotations and reflection do not!

Exercise 6.2. Every $g \in \text{Isom}(\mathbb{E}^2)$ is one of these five types.

Recall: If F is a complete metric space then we say F has a*geometry modelled on X* if for all $x \in F$ there are open neighbourhoods $U \subset F$ and $V \subset X$ so that $x \in U$ and so that U and V are isometric.

Theorem 6.3. *If F is a manifold modelled on X then there is a discrete group $G <$ Isom(X) acting freely such that* $X/G \underset{isom}{\cong} F$.

Proof. Hint: G arises as the deck group of a covering $\pi\colon X \to F$. $\square$

Definition 6.4. Suppose $\pi\colon M \to N$ is a cover. Then the *deck group* of π is $\mathrm{Deck}(\pi) = \{g \in \mathrm{Homeo}(M) \mid \pi \circ g = \pi\}$.

FIGURE

So g can only move p up or down, hence g is an integer and $\mathrm{Deck}(\pi) = \mathbb{Z}$.

Exercise 6.5. Last time we claimed that
$\mathrm{Isom}(\mathbb{E}^1) \supset < R_0, R_{\frac{1}{2}} > \cong D_\infty = < \alpha, \beta \mid \alpha^2, \beta^2 >$.

The outline of a way to show this was to define $\Phi\colon D_\infty \to < R_0, R_{\frac{1}{2}} >$ by $\alpha \mapsto R_0$ and $\beta \mapsto R_{\frac{1}{2}}$ and in general $\Phi(\omega(\alpha, \beta)) = \omega(R_0, R_{\frac{1}{2}})$ for a word ω. Therefore it needs to be shown that:

- The mapping is a group homomorphism.
- The mapping is a surjection.
- Finally that the mapping is injective, that is $\ker(\Phi) = \{\epsilon\}$.

To prove the third part use the four normal forms for elements in D_∞, namely $(\alpha\beta)^n\alpha, (\beta\alpha)^n\beta, (\alpha\beta)^n$ and $(\alpha\beta)^n$ and a *tiling* of $\mathbb{R}$.

FIGURE

If G acts in a space X then call $P \subseteq X$ a fundamental domain if

- P tiles X, that is $\bigcup_{g\in G} g(P) = X$.
- We have $\mathrm{int}(P) \cap \mathrm{int}(g(P)) = \varnothing$.
- "P is a nice cell," that is $P \cong \mathbb{B}^n$.