Saul Schleimer
Andrea Nagy MA4J2
17th Jan 2011

4 Lecture 4

Google the course notes by Thurston, Conway, Doyle, Gilman: "Geometry and the imagination".

Definition 4.1. $\mathbb{E}^n = (\mathbb{R}^n, ds_{\mathbb{E}})$ where $ds_{\mathbb{E}}$ is the standard length element on $\mathbb{R}^n$, so $ds_{\mathbb{E}}^2 = dx_1^2 + dx_2^2 + ... + dx_n^2$.

Definition 4.2. If $\gamma : [a, b] \to \mathbb{R}^n$ is a smooth path with $t \mapsto (\gamma_1(t), \gamma_2(t), ..., \gamma_n(t))$, then

$$\int_\gamma ds_{\mathbb{E}} = \int_a^b \sqrt{(\gamma_1')^2 + (\gamma_2')^2 + ... + (\gamma_n')^2}\,\mathrm{d}t$$

is the *length* of γ.

Example 4.3. Let $\gamma(t) = (\cos t, \sin t)$ for $t \in [0, \pi]$. Then

$$\int_\gamma ds = \int_0^\pi \sqrt{(-\sin t)^2 + (\cos t)^2}\,\mathrm{d}t = \int_0^\pi \mathrm{d}t = \pi$$

In general, if M is a manifold and ds_M is a length element, then for any $p, q \in M$ define $d_M(p, q) = \inf\{\int_\gamma ds_M \mid \gamma : [0, 1] \to M, \gamma(0) = p, \gamma(1) = q, \gamma \text{ is smooth}\}$.

Exercise 4.4. d_M is a metric. Need to check that it is reflexive, symmetric and that the triangle inequality holds.

Example 4.5. Consider $(\mathbb{R}^2, \frac{2ds_{\mathbb{E}}}{1+r^2})$ where $r^2 = x^2 + y^2$. Then this new metric agrees with $ds_{\mathbb{E}}$ on the unit circle. So if $\gamma : [a, b] \to \mathbb{R}^2$ is a path then write $\gamma(t) = (\alpha(t), \beta(t))$. Now define $ds_{\mathbb{S}} = \frac{2ds_{\mathbb{E}}}{1+r^2}$ and so

$$\int_\gamma ds_{\mathbb{S}} = \int_\gamma \frac{2ds_{\mathbb{E}}}{1 + r^2} = \int_a^b \frac{2\sqrt{(\alpha')^2 + (\beta')^2}}{1 + \alpha^2 + \beta^2}\,\mathrm{d}t$$

Exercise 4.6.

- $(\mathbb{R}^2, ds_{\mathbb{S}})$ is not complete.
- [Harder] $(\mathbb{R}^2, ds_{\mathbb{S}})$ is locally homogeneous.
- $(\mathbb{R}^2 \setminus \{0\}, \frac{ds_{\mathbb{E}}}{r})$ is complete and homogeneous.

In dimension 2, suppose $f(x, y) = (u, v)$. So we may write du in terms of x, y, dx, dy and similarly for dv.

Example 4.7. Let rotation through angle θ be given by $R_\theta(x, y) = (\cos\theta x - \sin\theta y, \sin\theta x + \cos\theta y)$, where θ is a constant. Now $du = \cos\theta dx - \sin\theta dy$ and $dv = \sin\theta dx + \cos\theta dy$. To check if R_θ is an isomtery of ds_E, compute

$$du^2+dv^2 = \cos^2\theta dx^2+\sin^2\theta dy^2-2\cos\theta\sin\theta dxdy+\sin^2\theta dx^2+\cos^2\theta dy^2+2\cos\theta\sin\theta dxdy = dx^2+dy^2 = ds_E$$

Hence R_θ is an isometry.

Remark 4.8.

- R_θ is a rotation
- $T_(a, b)(x, y) = (x + a, y + b)$ is a translation
- $F(x, y) = (x, -y)$ is a reflection

Hence the conclusion is that $\mathbb{R}^2 \rtimes O(2) \subseteq \text{Isom}(\mathbb{E}^2)$. But we want to show that this is an equality. For this we need a theorem.

Theorem 4.9. *A straight line is the shortest distance between two points.*

Example 4.10. Is the infimum always realised? The answer is no, for example if M is not complete. Consider $(\mathbb{R}^2 \setminus 0, ds_{\mathbb{E}})$ with points $P = (-1, 0), Q = (1, 0)$.

Exercise 4.11. Show that if (M, ds) is complete, then the infimum is attained and is a minimum, where M is a smooth manifold. If this is false, add hypothese until it is true.

Proof of Theorem. Suppose $P = (0, 0)$ and $Q = (0, q)$ are on the y-axis. Suppose that $\gamma : [0, 1] \to \mathbb{R}^2$ as $\gamma(t) = (\alpha(t), \beta(t))$ with $\gamma(0) = P, \gamma(1) = Q$. Then

$$\int_\gamma ds \ = \int_0^1 \sqrt{(\alpha')^2 + (\beta')^2}\,\mathrm{d}t \geq \int_0^1 \sqrt{0 + (\beta')^2}\,\mathrm{d}t$$

Define $\delta(t) = (o, \beta(t))$. So "covering up α" means that we think of γ as δ. Hence $\int_0^1 \sqrt{0 + (\beta')^2}\,\mathrm{d}t = \int_\delta ds$. So the length of γ is at least the length of δ, which in turn is *at least the length of* $[P, Q]$, the line segment.

For general P, Q, there is an isometry sending $P \mapsto Q$ and $Q \mapsto Q - P$ (translation), and then fixing th origin and moving $Q-P$ to the y-axis (rotation). Since translations and rotations preserve straight lines, we are done. ☐

So straight lines in $\mathbb{E}^2$ globally minimise length.

Definition 4.12. $L \subset (M, ds)$ is a *geodisc* is L locally minimses length (L is 1-dimensional).

Example 4.13. Going anticlockwise around the unit circle minimses the length locally.

Exercise 4.14.
- $\mathbb{R}^2 \rtimes O(2) = \text{Isom}(\mathbb{E}^2)$
- $\mathbb{L} = (\mathbb{R}^2 \setminus 0, \frac{ds_{\mathbb{E}}}{r})$ and $\text{Isom}(\mathbb{L}) = (\mathbb{R} \rtimes O(2)) \rtimes \frac{\mathbb{Z}}{2\mathbb{Z}}$