Three Manifolds - Lecture 3

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Theorem 1 (Singer's Theorem)**.** If a metric space M is complete, locally homogeneous, and simply connected, then M is homogeneous, (i.e the group Isom(X) acts transitively on the space X).

3.0.1 Examples of Universal covers

Reference: See Hatcher: Algebraic Topology

A simply connected cover of a space X is the universal cover. So, if a space is simply connected then it is its own universal cover, (see Example 1). We only allow connected covers.

1. $\mathbb{S}^n$ is its own universal cover iff $n \neq 1$.
2. $\mathbb{R}^2 \mapsto \mathbb{T}^2 \quad (x, y) \mapsto (x \mod 1, y \mod 1)$
3. $\mathbb{R} \mapsto \mathbb{S}^1 \quad t \mapsto (\cos(t), \sin(t))$

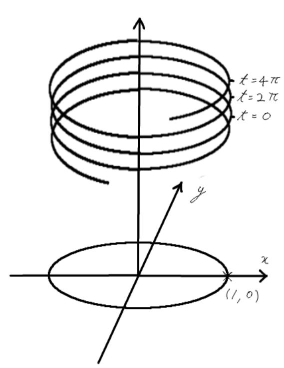

Figure 1: $\mathbb{R}^1$ is the universal cover of $\mathbb{S}^1$

Exercise 1. Show that completeness is a necessary hypothesis to Singer's Theorem by means of an example, i.e find a locally homogeneous, simply connected space, which is not homogeneous.

Definition 1 (Geometry)**.** If a space X is complete, simply connected and homogeneous then we call the pair $(X, Isom(X))$ a geometry. (We assume a metric associated with the space X).

Example 1 (Euclidean Geometry). $\mathbb{E}^2$ is the geometry $(\mathbb{R}^2,\ \mathbb{R}^2 \rtimes O(2))$ where O(2) is the orthogonal group of matrices and $\rtimes$ is the semidirect product.

More concretely, $\mathbb{E}^2 = (\mathbb{C},\ \ \mathbb{C} \rtimes O(2))$.

A general element $f \in Isom(\mathbb{C})$ has the form

$z \mapsto e^{i\theta} z + a$ (orientation preserving)

$z \mapsto e^{i\theta} \bar{z} + a$ (orientation reversing)

Call $T_a z = z + a$ a translation and call $R_\theta(z) = e^{i\theta} z$ a rotation. So O(2) acts on $\mathbb{C}$ by rotation.

Exercise 2. Check that $R_\theta \circ T_a \circ R_\theta^{-1} = z + e^{i\theta} a$

Recall: A semidirect product of two subgroups A and B, (where A will be a normal subgroup), is denoted by $A \rtimes_\phi B$. It is the group on the set $A \times B$ with multiplication

$$(a, b)(a', b') = (a \cdot \phi_b(a'), b \cdot b')$$

where $\phi : B \mapsto Aut(A)$ is a homeomorphism.

Once more: If

$$f(z) = e^{i\theta} z + a, \quad g(z) = e^{i\phi} z + b$$

then

$$f \circ g(z) = f(e^{i\phi} z + b) = e^{i\theta} e^{i\phi} + e^{i\theta} b + a = e^{i(\phi+\theta)}$$

Definition 2. Fix a geometry $(X, Isom(X))$ with a complete metric such that $\forall x \in F$, there exists a neighbourhood U of x in F and a neighbourhood $V \subseteq X$ such that U is isometric to V. Then we say F has a geometric structure modelled on X.

Example 2. The infinite Mobius band is not homogeneous. The centre line has half the length of any line parallel to it, due to the identifications made. So there is no isometry which takes a point on the centre line to a point which is not on the centre line.

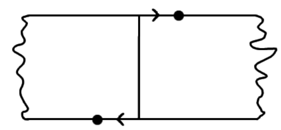

Figure 2: The infinite Mobius band

$$\mathbb{M} = I \times \mathbb{R}/\sim \qquad \text{where} \qquad (1, y) \sim (0, -y)$$

is modelled on the geometric structure $\mathbb{E}^2$, but is not a geometric structure itself.

Here is the classification of 2-dimensional geometries.

X	Isom(X)	Curvature	Local Picture	X/Γ	$\chi(X/\Gamma)$
$\mathbb{S}^2$	O(3)	+1	cap/cup	$\mathbb{S}, \mathbb{P}$	> 0
$\mathbb{E}^2$	$(\mathbb{R}^2, \mathbb{R}^2 \rtimes O(2))$	0	flat	$\mathbb{T}^1, \mathbb{K}$	$= 0$
$\mathbb{H}^2$	$\mathbb{P}GL(2, \mathbb{R})$	-1	saddle	Everything else	< 0

<u>Goal</u>: Thurston classified the possible geometries in dimension 3. They are:

- Products: $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{E}^3 = (\mathbb{E}^2 \times \mathbb{E})$, $\mathbb{H} \times \mathbb{R}$
- Constant curvature: $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$
- Twisted: *Nil*, *PSL*, *Solv*