17 Lecture 17

Continuing our discussion from last time, define

$$K = I \times \mathbb{D}^2/(1, z)\ (0, \bar{z}).$$

Note we can orient the circles of K by increasing direction of the I factor, but K itself is not orientable. This is is a higher dimensional version of the Möbius band.

$$\mathbb{M} = I \times B_1/(1, t)\ (0, -t) \quad \text{where} \quad B^1 = [-1, 1]$$

Call K the solid Klein Bottle. Note that $\partial K = \mathbb{K}^2$ and that $T(p, q)$ and K are p-fold covered and 2-fold covered respectively by $T(1, 0)$, by a map sending circles to circles.

Definition 17.1. We say $M^3 = \sqcup S^1$ equipped with partition into circles is a *Seifert fibred space* if for every circle C in the partition, there exists an open neighbourhood $U \subset M$ of C such that U is homeomorphic (preserving cricles) to either $T(p, q)$ or K.

Definition 17.2. We call the circles of the partition of M *fibres.* We say that Seifert spaces M and N are isomorphic if there is a homeomorphism $f : M \to N$ that send fibres to fibres.

Exercise 17.3. Show that $T(p, q)$ is isomorphic to $T(p', q')$ if and only if $p = p'$ and $q = q'$ (or $q = \pm q'\ (modp)$ if we allow orientation to be reversed).

So q/p is a complete invarient for $T(p, q)$.

Remark 17.4. If $C \subseteq \partial M$ then any small fibred neighbourhood for C is isomorphic to

$$I \times \mathbb{D}^2/(1, z)\ (0, z).$$

Exercise 17.5. Suppose F^2 is a surface, then $S^1 \times F$ is a Seifert fibred surface with partition

$$\{S^1 \times \{x\} : x \in F\}.$$

We claim that if F has genus greater than or equal to 2, then this is the unique Seifert fibred structure on $M = S^1 \times F$ (up to isomorphism).

Remark 17.6. This does not hold when $M = S^1 \times S^2$ ie. $F = S^2$.

Exercise 17.7 (Difficult)**.** $\mathbb{R}^3$ can be partitioned into circles but $\mathbb{R}^3$ does not admit a Seifert fibred structure.

Definition 17.8. If $C \subset M$ has a neighbourhood isomorphic to $T(1, 0)$ we call C a *regular fibre.* If not call C a *critical fibre.* Finally, if all sufficiently small neighbourhoods of C are isomorphic to $T(p, q)$ we say q/p is the *invariant of* C.

Note that M^3 and all fibres must be oriented to define the invariant. Note also that $S^1 \times F$ has no critical fibres.

Example 17.9. Let

$$\mathbb{T}^2 = I^2/(x, 0)\ (x, 1), (0, y)\ (1, y).$$

Recall $h\colon \mathbb{T}^2 \to \mathbb{T}^2$ was given by $(x, y) \mapsto (1 - x, 1 - y)$, so define

$$M = I \times \mathbb{T}^2/(1, z)\ (1, h(z)).$$

We claim that M has four critical fibres. Consider the torus $\{\frac{1}{2} \times \mathbb{T}^2.\}$ This can be divided as follows. Consider the planes $x = \frac{1}{4}$ $y = \frac{1}{4}$ and $x = \frac{3}{4}$ $y = \frac{3}{4}$. These cut $M = I \times \mathbb{T}^2/h$ into four copies of $T(2, 1)$.

Remark 17.10. We may also fibre $M = I \times \mathbb{T}^2/h$ by horizontal circles all parallel ot the x-axis. This Seifert fibred structure has no critical fibres, hence the are not isomorphic.