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16 Lecture 16

An *orbifold* (roughly) is a topological space F such that for all $x \in F$ there is a neighbourhood $U \subseteq F$ modelled on $\mathbb{R}^2_{(+)}/G_x$ and we call G_x the *local group* at x.

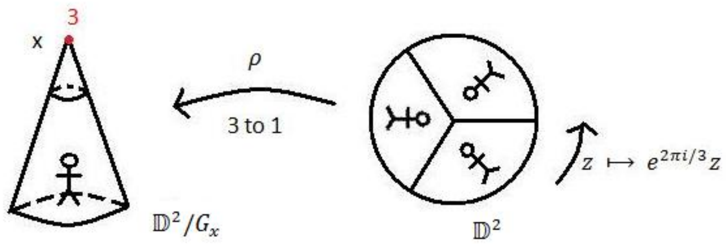

Figure 1: Example of an orbifold.

Example 16.1. Recall, if a group G acts on a set X, then $\mathrm{Stab}(x) = \{g \in G \mid g \cdot x = x\}$.

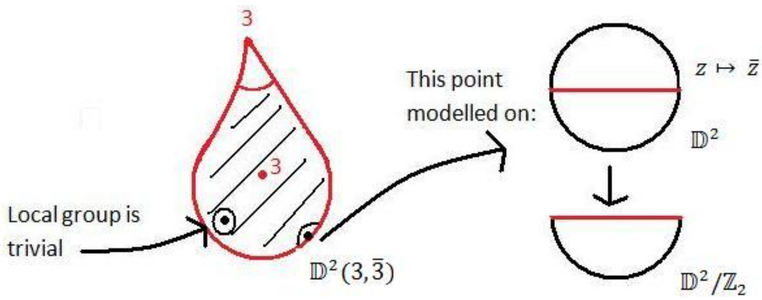

Figure 2: Examples of local groups for points in $\mathbb{D}^2(3, \bar{3})$.

For the corner reflector the local group is the dihedral group, D_6.

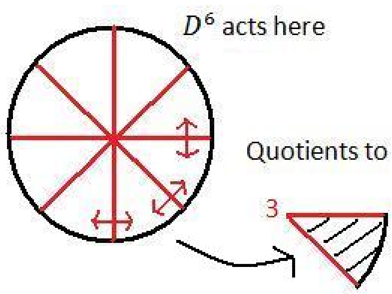

Figure 3: Local group of the corner reflector in $\mathbb{D}^2(3, \bar{3})$.

Exercise 16.2. $\mathbb{D}^2(3, \bar{3})$ has a six-fold cover that is a surface.

Example 16.3. Consider $\mathbb{T}^2 = I^2/(x, 1) \sim (x, 0), (1, y) \sim (0, y)$.

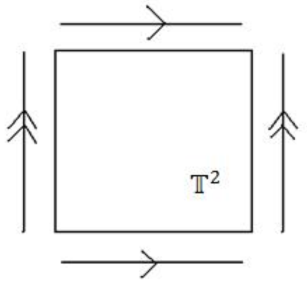

Figure 4: $\mathbb{T}^2$.

Define $h : \mathbb{T}^2 \to \mathbb{T}^2$, $(x, y) \mapsto (1 - x, 1 - y)$.
Note that $\text{fix}(h) = \{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0), (0, \frac{1}{2}), (0, 0)\}$.

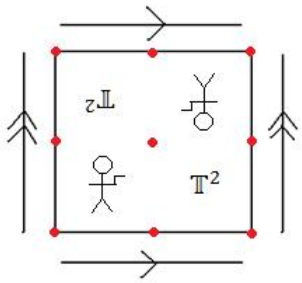

Figure 5: How h affects T.

Note that h, near any of the fixed points, is a rotation by 180^o.

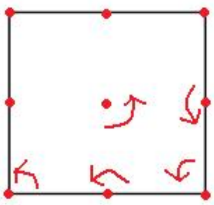

Figure 6: h, near any of the fixed points, is a rotation by 180^o.

Define $F = \mathbb{T}^2/\langle h \rangle$.

Definition 16.4. An *isomorphism* of orbifolds is a homeomorphism of the underlying topological spaces that preserves the orbifold structure.

Exercise 16.5. Show $F = \mathbb{T}^2/\langle h \rangle$ is isomorphic $S^2(2, 2, 2, 2)$.

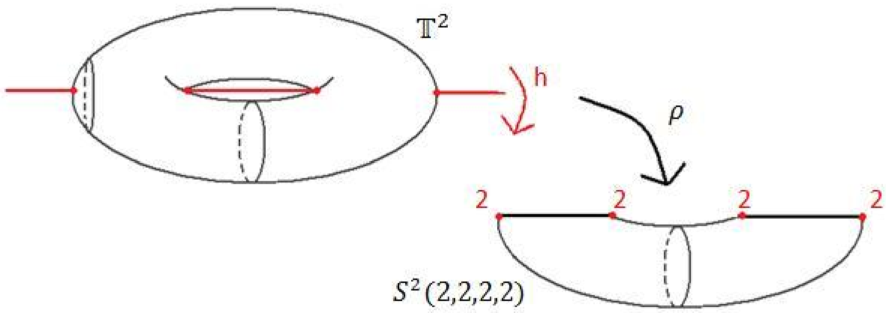

Figure 7: Image of $\mathbb{T}^2$ and $\mathbb{T}^2/\langle h\rangle$.

Exercise 16.6. Consider, instead of h, the map $r : \mathbb{T}^2 \to \mathbb{T}^2$, $(x, y) \mapsto (1-y, x)$. Compute $\text{fix}(r)$ and the orbifold $\mathbb{T}^2/\langle r\rangle$.

Exercise 16.7. Show that $\mathbb{D}^2(p, \bar{q})$ is good by finding a surface that finitely covers $\mathbb{D}^2(p, \bar{q})$ [Restrict to $p = 2$ if you like].

Definition 16.8. A *Seifert fibered space* is a three-manifold equipped with a "nice" partition into circles

$$M = \sqcup S^1.$$

Definition 16.9. We call the circles of the partition *fibers*.

Exercise 16.10. Consider F^2, a surface. Let $M = S^1 \times F^2$. The fibers have the form $S^1 \times \{x\}$ for $x \in F^2$. Notice that the projection $\rho_F : M \to F^2, (e^{i\theta}, x) \mapsto x$ is a *bundle map*. [Reading exercise]

Example 16.11. We define the *fibered solid torus* as follows. Suppose $p, q \in \mathbb{Z}$ so that $1 \leq q \leq p$ and $\gcd(p, q) = 1$. Then $T(p, q) = I \times \mathbb{D}^2/(1, z) \sim (0, e^{2\pi i \frac{q}{p}} \cdot z)$

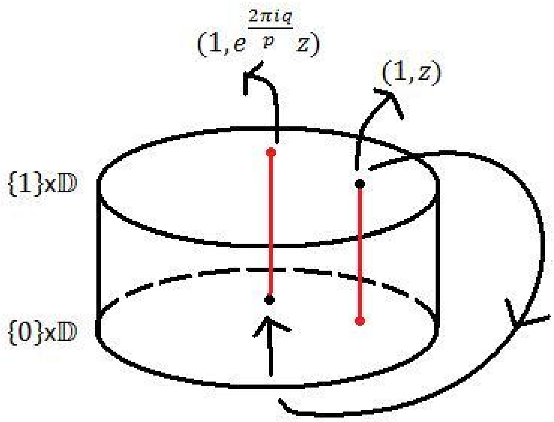

Figure 8: Image of $T(p, q)$.

Exercise 16.12. Show $T(p, q) \equiv T(p, p + q)$.

Note: $T(p, q)$ is equipped with a partition into circles coming from the vertical intervals.

We also equip $T(p, q)$ with the orientation coming from that of the disc, i.e counterclockwise and that of the interval oriented in the direction of increasing coordinate.

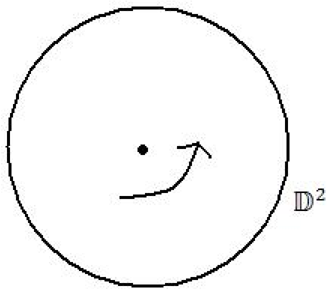

Figure 9: Orientation of the disc.

We also orient the circles of $T(p, q)$ using the orientation of I.

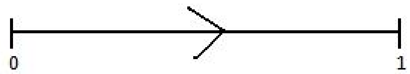

Figure 10: Orientation of I.

Exercise 16.13. Here is T(3,1).

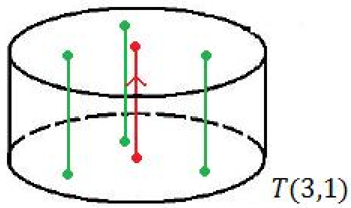

Figure 11: Image of T(3,1).

Note that here the centre closes up immediately.

Exercise 16.14. Equipped with these orientations $T(p, q)$ is isomorphic to $T(p', q')$ if and only if $p = p'$ and $q = q'$ [if we forget orientations $q = \pm q' (\operatorname{mod} p)$].