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14 Lecture 14

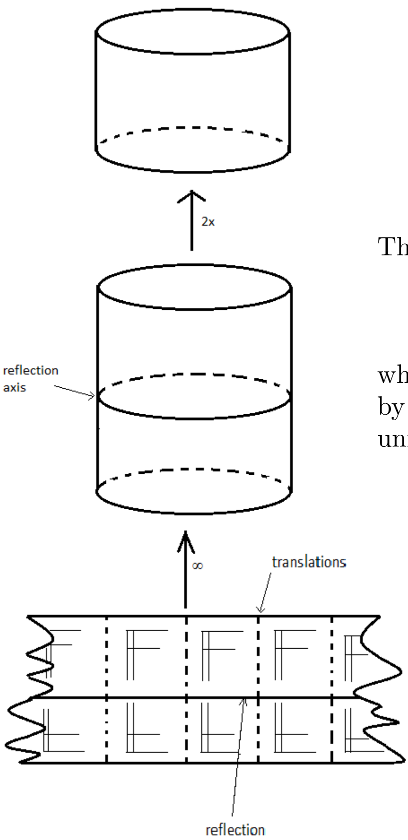
14.1 Two-orbifolds

Definition 14.1. If F is a connected 2-orbifold that is good, then we define

$$\pi_1^{orb}(F) = \text{Deck}(\rho\colon \tilde{F} \to F)$$

Where $\rho\colon \tilde{F} \to F$ is the universal cover.

Example 14.2. Consider $F = S^1 \times I(\overline{2})$



This is double covered by $S^1 \times I$

which in turn is infinitely covered by the strip $\mathbb{R} \times I$ so $\mathbb{R} \times I$ is the universal cover of $S^1 \times I(\overline{2})$

In terms of co-ordinates:

$$\begin{aligned} \rho\colon \mathbb{R} \times [-1, +1] &\to F \\ (t, x) &\to (e^{it}, |x|) \end{aligned}$$

So the deck group $\text{Deck}(\rho) \cong \mathbb{Z} \times \mathbb{Z}_2$.
So [Check] for any $g \in \text{Deck}(\rho)$, $\rho \circ g = \rho$.

Exercise 14.3. If $\rho\colon E \to F$ is an orbifold cover then

$$\pi_1^{\mathrm{orb}}(E) < \pi_1^{\mathrm{orb}}(F) \text{ and } \deg(\rho) = [\pi_1^{\mathrm{orb}}(F) : \pi_1^{\mathrm{orb}}(E)].$$

Example 14.4. Consider H, the half-mirrored hexagon,shown below:

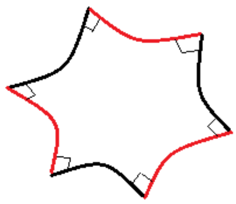


Figure 1: Half-mirrored hexagon, the red edges represent mirrored edges and the black edges represent the regular boundary

Claim 1. The orbifold fundamental group $\pi_1^{\mathrm{orb}}(H)$ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \cong \langle a, b, c \mid a^2, b^2, c^2 \rangle$

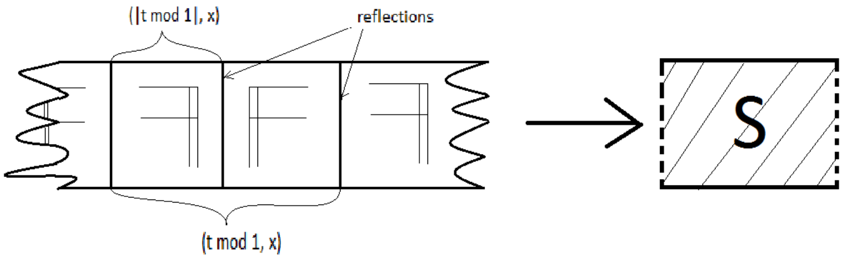
Example 14.5. Consider S, the half-mirrored square, shown below:

Figure 2: Half-mirrored square, the red edges represent mirrored edges and the black edges represent the regular boundary

Claim 2. The orbifold fundamental group $\pi_1^{\mathrm{orb}}(S)$ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$ which is isomorphic to $\mathbb{D}_\infty \cong \langle a, b, \mid a^2, b^2 \rangle$

Proof. We have the following covering map:

$$\rho\colon \mathbb{R} \times I \to S$$
$$(t, x) \to (\mid t \bmod 1 \mid, x)$$

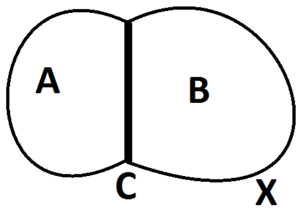


Q.E.D

Scott tells us that the orbifold fundamental group can be computed directly via the Seifert-van-Kampen Theorem

Theorem 14.6 (Seifert-van-Kampen)**.** *Suppose $X = A \cup B$ and $C = A \cap B$ are path connected spaces. Choose $x \in C$ to be the base point for all computations. Let $f\colon C \to A$ and $g\colon C \to B$ be the inclusion maps.*
Then:

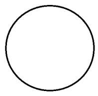
$$\pi_1(X) \cong \frac{\pi_1(A) * \pi_1(B)}{\langle\langle f_*(\omega) \cdot g_*(\omega^{-1}) \ \mid \ \omega \in \pi_1(c)\rangle\rangle}$$



Note: We are really imposing the new relation $f_*(\omega) = g_*(\omega)$.

In the same way, we can decompose F(a 2-orbifold) along a compact 1-orbifold and compute fundamental groups. We first review the list of compact 1-orbifolds.

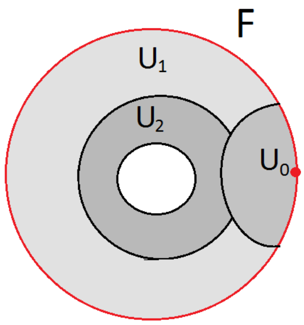
Review:

X :	S^1	I	$I(\bar{2})$	$I(\bar{2},\bar{2})$
Picture:				
$\pi_1^{\text{orb}}(X)$:	$\mathbb{Z}$	$\mathbb{1}$	$\mathbb{Z}_2$	D_∞
$\tilde{X}$(universal cover) :	$\mathbb{R}^1$	I	I	$\mathbb{R}$

Recall: The infinite dihedral group D_∞ has presentation $\langle a, b \mid a^2, b^2 \rangle$ and is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$

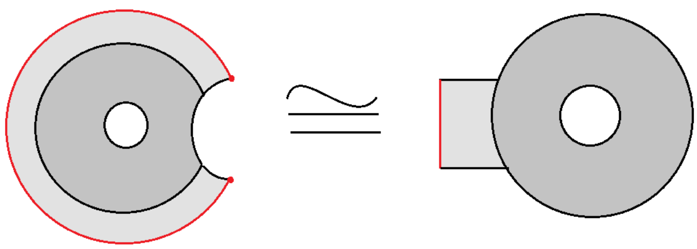
14.2 Computations using Seifert-van-Kampen

Example 14.7. Let $F = S^1 \times I(\bar{2})$ We place a vertex on the mirror boundary. Let U_0 be a small neighbourhood of the vertex. Let U_1 be a small neighbourhood of the mirror boundary. Define $U_2 = F \min(U_0 \cup U_1)$



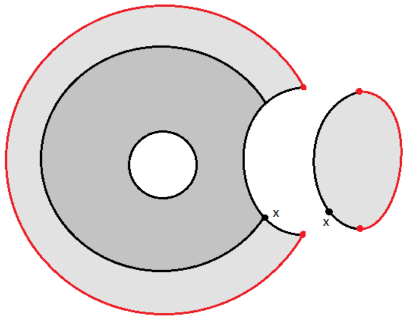
$$\pi_1^{\text{orb}}(U_0) = \mathbb{Z}_2 \qquad (U_0 \cong I \times I(\bar{2}))$$
$$\pi_1^{\text{orb}}(U_1) = \mathbb{Z}_2 \qquad (U_1 \cong U_2)$$
$$\pi_1^{\text{orb}}(U_2) = \mathbb{Z} \qquad (U_2 \text{ is an annulus})$$

Note that $U_2 \cap U_1 \cong I$ so $\pi_1^{\text{orb}}(U_2 \cup U_1) = \mathbb{Z} * \mathbb{Z}_2$



Fixing, notation we write $\pi_1^{\text{orb}}(U_2 \cup U_1) \cong \langle t, r \mid r^2 \rangle$

Since $U_0 \cap (U_1 \cap U_2) \cong I(\overline{2}, \overline{2})$ we find that $\pi_1^{\text{orb}}(F) \cong (\mathbb{Z} * \mathbb{Z}_2) \underset{D_\infty}{*} \mathbb{Z}_2$



Fixing notation, we write $D_\infty = \langle a, b \mid a^2, b^2 \rangle$ and $\mathbb{Z} * \mathbb{Z}_2 = \langle t, r \mid r^2 \rangle$. Let $\mathbb{Z}_2 = \langle s \mid s^2 \rangle$. Then the two inclusion maps f,g give:

$$\begin{aligned} f_\star \colon a &\to r \\ b &\to trt^{-1} \\ g_\star \colon a &\to s \\ b &\to s. \end{aligned}$$

Hence, $\pi_1^{orb}(F) = \langle r, s, t \mid r^2, s^2, t^2 \rangle \cong \langle r, t \mid r^2, r^{-1}trt^{-1} \rangle \cong \mathbb{Z} * \mathbb{Z}_2$. This agrees with the orbifold fundamental group computed above.