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11 Lecture 11

Last time we began a discussion of the geometry of $\mathbb{H}^2$.

Exercise 11.1.

1. Read Scott's treatment of $\mathbb{H}^2$.
2. Carry out details of the plan given in Lecture 10; as done for $\mathbb{S}^2$, classify the isometries and geodesics for $\mathbb{H}^2$. Show that $\text{Isom}^+(\mathbb{H}^2) \cong PSL(2, \mathbb{R}^2)$, where $\text{Isom}^+(\mathbb{H}^2)$ are the orientation preserving isometries of $\mathbb{H}^2$.
3. Show that $(\mathbb{H}, \frac{ds}{y})$ is isometric to $(\mathbb{D}, \frac{2ds}{1-r^2})$.

Also see chapters one and two in Professor Series' notes for MA448 (Hyperbolic Geometry).

Recall: A 2-orbifold is a "nice" topological space locally modelled on $\mathbb{R}^2/G$, where G is a discrete subgroup of $O(2)$. We call G the *local group*.

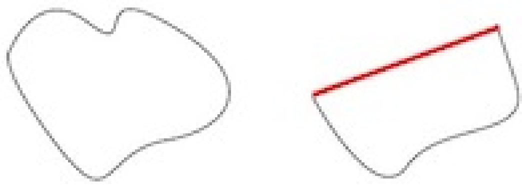

Figure 1: Local pictures for orbifolds: $\mathbb{R}^2/\langle 1\rangle$ and $\mathbb{R}^2/\langle\text{reflection}\rangle$

Figure 2: $\mathbb{R}^2/C_n$ with cone point of order n and $\mathbb{R}^2/D_{2n}$ with a corner reflection of order n

Figure 3: $\mathbb{R}^2_+/\langle 1\rangle$ and $\mathbb{R}^2_+/\mathbb{Z}_2$

See page 442 of Scott's article for a formal definition of orbifold, including the overlap condition.

Example 11.2. $\mathbb{S}^2(p, q, r)$ is the orbifold with underlying surface homeomorphic to $\mathbb{S}^2$ and having three cone points of order p, q and r respectively.

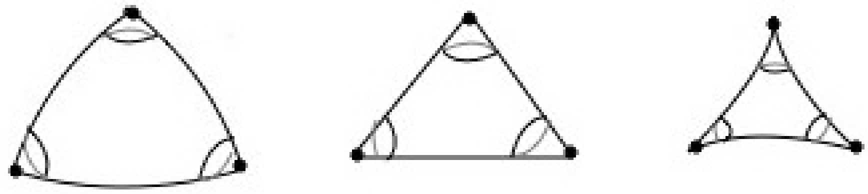

Figure 4: $\mathbb{S}^2$ with cone points of order p, q and r

11.1 Orbifold Euler Characteristic

We wish to use the local group G giving the model to define a new Euler characteristic.

Definition 11.3. Given F an orbifold, triangulate F so that cone points and corners are among the vertices and so that the mirror boundary lies in the 1-skeleton. We decree the following:

1. A vertex is worth $+1$.
2. Cone points of order n are worth $\frac{1}{n}$.
3. Corner reflectors of order n are worth $\frac{1}{2n}$.
4. Half-mirrored corners are worth $\frac{1}{2}$.
5. Mirrored edges are worth $-\frac{1}{2}$.
6. Faces are worth $+1$.

Example 11.4.

$$\chi^{\text{orb}}(\mathbb{S}^2(p,q,r)) = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 3 + 2 = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1$$

Figure 5: $\mathbb{S}^2(p,q,r)$

Example 11.5.

$$\chi^{\text{orb}}(\mathbb{D}^2(\bar{p},\bar{q},\bar{r})) = \frac{1}{2p}+\frac{1}{2q}+\frac{1}{2r}-\frac{3}{2}+1 = \frac{1}{2}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)-\frac{1}{2} = \frac{1}{2}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1\right)$$

Two copies of $\mathbb{D}^2(\bar{p},\bar{q},\bar{r})$ glued along their boundary give $\mathbb{S}^2(p,q,r)$.

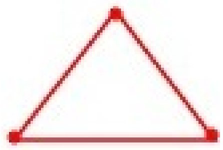

Figure 6: $\mathbb{D}^2(\bar{p}, \bar{q}, \bar{r})$)

Remark 11.6. There is a degree two *orbifold covering map*

$$\mathbb{S}^2(p,q,r) \xrightarrow{\times 2} \mathbb{D}^2(\bar{p}, \bar{q}, \bar{r})$$

so we expect that $\chi^{\text{orb}}(\mathbb{S}^2(p,q,r)) = 2 \cdot \chi^{\text{orb}}(\mathbb{D}^2(\bar{p}, \bar{q}, \bar{r}))$.

Exercise 11.7. Classify all connected 2-orbifolds F with $\chi^{\text{orb}}(F) > 0$. (More: do the same for $\chi^{\text{orb}}(F) = 0$. In both cases you may restrict to orbifolds without boundary and/or mirror boundary.)

Remark 11.8. $\chi^{\text{orb}}(\mathbb{S}^2(p,q,r))$ has sign

1. positive $\iff \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.
2. zero $\iff \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.
3. negative $\iff \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$.

This corresponds to the fact that triangles in $\mathbb{S}^2$, $\mathbb{E}^2$ and $\mathbb{H}^2$ have angle sum greater than, equal to and less than π respectively.

Exercise 11.9. Define, for $x \in F$, $\text{ord}(x) =$ order of the point x. Show that if F has no mirrors,

$$\chi^{\text{orb}}(F) = \chi(|F|) - \sum_{x \in F} \left(1 - \frac{1}{\text{ord}(x)}\right)$$

where $\chi(|F|)$ is the usual Euler characteristic of the underlying space, forgetting the orbifold structure.

11.2 Orbifold Covers

Suppose that $H \lneq G < O(2)$. Then there is a quotient map

$$\begin{array}{ccc} H \cdot x & \in & \mathbb{R}^2/H \\ \downarrow & & \downarrow \\ G \cdot x & \in & \mathbb{R}^2/G \end{array}$$

Exercise 11.10. Check that this is well defined.

Figure 7 shows the lattice of two-fold covers for $G = D_8 = \text{Sym}(\square)$:

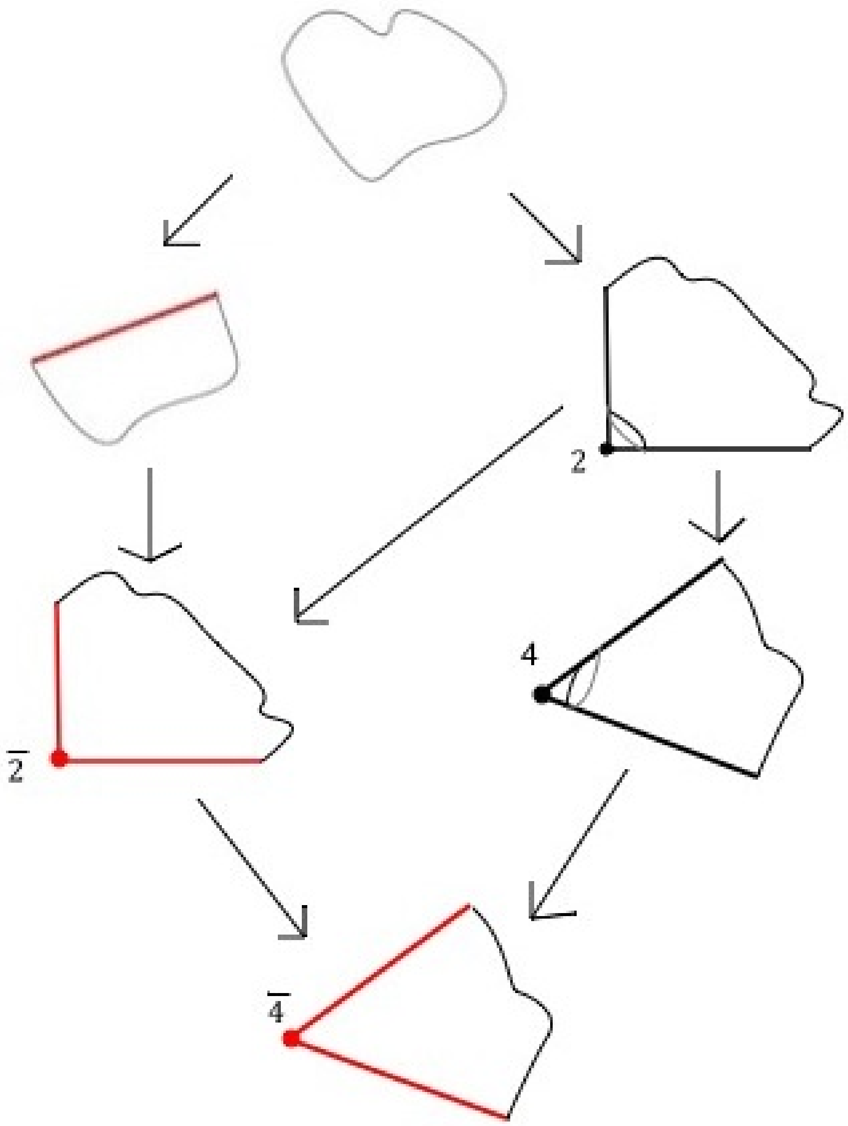

Figure 7: Orbifold covers for the group D_8. All covers shown have degree two.