Please let me know if any of the problems are unclear or have typos.

Exercise 4.1.

- Classify the discrete subgroups of $\mathrm{Isom}(\mathbb{S}^1) = O(2)$.
- A discrete subgroup $G < \mathrm{Isom}(\mathbb{E}^2)$ is a *point group* if G has a global fixed point. Using the fact $\mathrm{Isom}(\mathbb{E}^2) = \mathbb{R}^2 \rtimes O(2)$ show all point groups are conjugate to one of the groups found above.

Exercise 4.2. [Medium] A discrete group $G < \mathrm{Isom}(\mathbb{E}^2)$ is a *frieze group* if G has no global fixed point, but does have an invariant line L. Classify the frieze groups up to conjugation and scaling.

Exercise 4.3. A discrete group $G < \mathrm{Isom}(\mathbb{E}^2)$ acting cocompactly is called a *wallpaper group*. Picking a fundmental domain gives a *wallpaper tiling*. Find and photograph a couple of point, frieze, or wallpaper tilings on campus with yourself or a friend in the frame. Identify the quotient orbifolds. Send me the photos and I will post them on the class website.

Exercise 4.4. [Easy - do not turn in.] Show $G < \mathrm{Isom}(X)$ is discrete if and only if there is an open subset $U \subset \mathrm{Isom}(X)$ so that for all $g \in G$ we have $(g \cdot U) \cap G = \{g\}$.

Exercise 4.5. Suppose $(X, \mathrm{Isom}(X))$ is a geometry. Suppose $G < \mathrm{Isom}(X)$ is discrete and acts freely on X. Show X/G is a manifold and the quotient map $\rho\colon X \to X/G$ is a covering map with deck group G.

Exercise 4.6. Suppose that P, Q, R are three non-collinear points of $\mathbb{E}^2$. Let α, β, γ be the three π–rotations about P, Q, R. Show, by giving a fundamental domain or otherwise, that $G = \langle \alpha, \beta, \gamma \rangle$ is a discrete subgroup of $\mathrm{Isom}(\mathbb{E}^2)$. [This was sketched in class. Please provide the missing details or provide a different solution.]

Exercise 4.7. Show that the group G of Exercise 4.6 is isomorphic to the abstract group

$$\langle \alpha, \beta, \gamma, \delta \mid \alpha^2, \beta^2, \gamma^2, \delta^2, \alpha\beta\gamma\delta \rangle.$$

Exercise 4.8. [Scott, Figure 1.4] Let T be an equilateral triangle in $\mathbb{E}^2$. Let α, β, γ be the three reflections in the three sides of T. Give a fundamental domain for $G = \langle \alpha, \beta, \gamma \rangle$. Show there is an index two subgroup $H < G$ that does not contain any reflections. Give a fundamental domain for H. [Optional: Find abstract presentations for G or for H.]