Please let me know if any of the problems are unclear or have typos.

Exercise 3.1. Recall that $ds^2 = dx^2 + dy^2$. In class we showed that $\mathbb{R}^2 \rtimes O(2) < \mathrm{Isom}(\mathbb{E}^2)$. We also showed that straight lines are geodesics. Prove $\mathbb{R}^2 \rtimes O(2) \cong \mathrm{Isom}(\mathbb{E}^2)$.

Exercise 3.2. Show great circles are geodesics for $\mathbb{S}^2$. Prove $\mathrm{Isom}(\mathbb{S}^2) = O(3)$.

Exercise 3.3. Let $ds_S = 2ds/(1 + r^2)$, where $r^2 = x^2 + y^2$. Consider the metric space $\mathbb{X} = (\mathbb{R}^2, ds_S)$. Prove $\mathbb{X}$ is not complete. [Optional challenge: Prove $\mathbb{X}$ is locally homogeneous.]

Exercise 3.4. Let $ds_L = ds/r$, with ds and r as above. Consider the metric space $\mathbb{L} = (\mathbb{R}^2 - \{0\}, ds_L)$. Prove from the definitions $\mathbb{L}$ is complete and homegeneous. [Optional challenge: Compute $\mathrm{Isom}(\mathbb{L})$. Classify geodesics in $\mathbb{L}$.]

Exercise 3.5. Here is another way to approach Exercise 3.4. With $\mathbb{L}$ as in that exercise, show the map $\rho\colon \mathbb{E}^2 \to \mathbb{L}$, defined by $\rho(x, y) = (e^x \cos y, e^x \sin y)$, is an isometric covering map. Deduce $\mathbb{L}$ is modelled on $\mathbb{E}^2$ and so is complete and (locally) homogeneous.

Exercise 3.6. Let $ds_D = 2ds/(1 - r^2)$. Consider the metric space $\mathbb{D} = (\mathbb{D}^2, ds_D)$ where $\mathbb{D}^2 \subset \mathbb{R}^2$ is the open unit disk. Prove $\mathbb{D}$ is complete and homogeneous.

Exercise 3.7. Suppose $(X, \mathrm{Isom}(X))$ is a geometry and $G < \mathrm{Isom}(X)$ is a subgroup. Prove G is discrete in $\mathrm{Isom}(X)$ if and only if G acts properly discontinuously on X.

Exercise 3.8. Show every isometry $f \in \mathrm{Isom}(\mathbb{E}^2)$ is either the identity, a reflection in a line, a translation, a rotation about a point, or a glide reflection along a line.

Exercise 3.9. Show every $f \in \mathrm{Isom}(\mathbb{S}^2)$ is a product of reflections in great circles. As in Exercise 3.8 give a classification of elements of $\mathrm{Isom}(\mathbb{S}^2)$. [Hint: it may be useful to first review the classification of elements of $\mathrm{Isom}(\mathbb{S}^1)$.]

Exercise 3.10. Let $R_a \in \mathrm{Isom}(\mathbb{E}^1)$ be the reflection $R_a(x) = 2a - x$. Give the details of the proof that $\langle R_0, R_1 \rangle < \mathrm{Isom}(\mathbb{E}^1)$ is isomorphic to the infinite dihedral group

$$D_\infty = \langle \alpha, \beta \mid \alpha^2, \beta^2 \rangle.$$