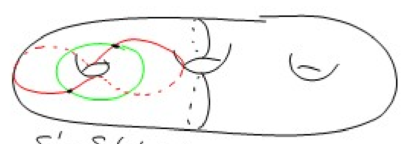
Lecture 17 The Combing

Building $P : \mathcal{C} \times \mathcal{C} \to \{\text{edge paths}\}$.

(2) If $d_S(\alpha,\beta) = 1$, $P(\alpha,\beta) = \{\alpha,\beta\}$

(1) If $d_S(\alpha,\beta) \geq 2$ Let $S' = S(\alpha,\beta)$ be the surface filled by α, β.



$S' = S(\alpha,\beta)$.

So $q_t^{\alpha\beta}$ is a sing. $\mathbb{E}^2$ metric on S'.

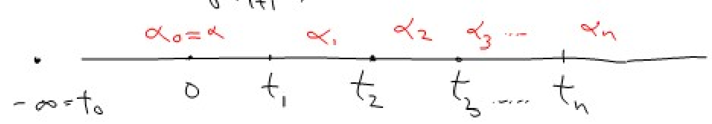
Let $\ell_t = \ell_{q_t^{\alpha\beta}}$. Set $\alpha_0 = \alpha$, $t_0 = -\infty$

Exercise: $\forall \alpha, \beta, \gamma$ the function $t \longmapsto \ell_t(\gamma)$ is convex.

Define $t_{i+1} = \max\{t \in \mathbb{R} \mid \ell_t(\alpha_{i+1}) \leq R\}$

[Recall $R \equiv 3R_0$]

Define: α_{i+1} is any widest curve in $S'(q_{t_{i+1}}^{\alpha\beta})$.



Exercise: This procedure terminates in a finite # of steps. In fact with $\alpha_n \in N_1(\beta) \subseteq \mathcal{C}(S)$.

By bounded diameter Lemma $d_S(\alpha_i, \alpha_{i+1}) \leq K$

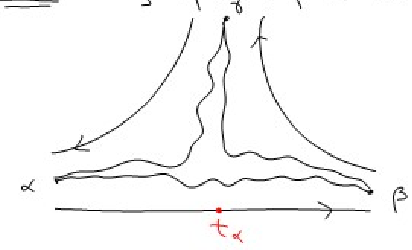
Define: $P(\alpha,\beta) = \left(\bigcup_{i=0}^{n-1} [\alpha_i, \alpha_{i+1}]\right) \cup [\alpha_n, \beta]$.

Rmk: Lots of choices! Not a symmetric construction. That is ok.

Easy Exercise: $P(\beta,\alpha) \subseteq N_{2K}(P(\alpha,\beta))$.

WTS: P is a $3K$-slim combing

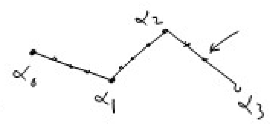
Case: $d_S(\alpha,\beta), d_S(\beta,\gamma), d_S(\gamma,\alpha) \geq 2$



Let i be the index s.t. $t_\alpha \in [t_i, t_{i+1}]$

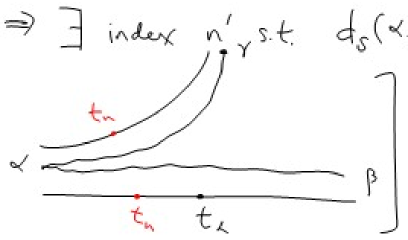
Define j, k similarly on the other sides $P(\beta,\gamma)$, $P(\gamma,\alpha)$.

Any vertex of $\mathbb{P}(\alpha,\beta)_{\leq i+1}$ is within K of some α_n $(n \leq i)$



α_n has width $\geq \varepsilon_0$ at time $t_n < t_\alpha \Rightarrow \alpha_n \in L(q_{t_n}^{\alpha\gamma}, R)$

$\Rightarrow \exists$ index n' s.t. $d_S(\alpha_n, \gamma_{n'}) \leq K$.



$$\left.\begin{array}{c} \alpha_n \in W(q_{t_n}^{\alpha\beta}, \varepsilon_0) \\ \cap\! \mid \\ \gamma_{n'} \in L(q_{t_n}^{\alpha\gamma}, R) \end{array}\right]$$

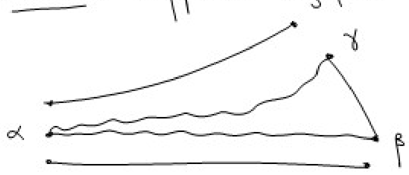
Recall that $q_t^{\alpha\gamma}$ is a $90°$ rotation of $q_{t'}^{\gamma\alpha}$.

$\therefore\ \mathbb{P}(\alpha,\beta)_{\leq i+1} \subseteq N_{2K}(\mathbb{P}(\gamma,\alpha))$

$\therefore\ \mathbb{P}(\alpha,\beta)_{\geq i} \subseteq N_{2K}(\mathbb{P}(\beta,\gamma))$.

So the triangle is $2K$ slim. ✓

<u>Case</u>: Suppose $d_S(\beta,\gamma) = 1$, others ≥ 2.



Let $\bar\beta = \frac{1}{i(\alpha,\beta)} \cdot \beta$

Let $\bar\gamma = \frac{1}{i(\alpha,\gamma)} \cdot \gamma$

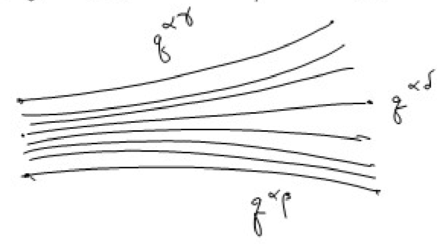
<u>Recall</u>: $q^{\alpha\beta} = q^{\alpha\bar\beta}$. Define $\delta = \frac{1}{2}(\bar\beta + \bar\gamma)$

<u>Note</u>: $i(\alpha,\bar\beta) = i(\alpha,\bar\gamma) = i(\alpha,\delta) = 1$.

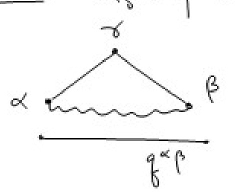
<u>Exercise</u> $\forall t$ $W(q_t^{\alpha\delta}, \varepsilon_0) \subseteq L(q_t^{\alpha\bar\beta}, R)$

[<u>Hint</u>: Review proof of $W \subseteq L$ part II (last time)]

<u>Consequence</u>: $\mathbb{P}(\alpha,\beta) \subseteq N_{3K}(\mathbb{P}(\alpha,\gamma))$. //



<u>Case</u>: $d_S(\alpha,\beta) \geq 2$, others at distance 1



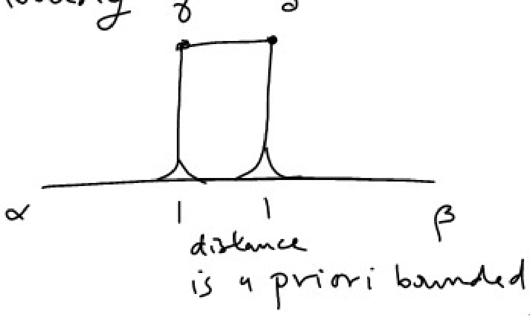
$\mathbb{P}(\alpha,\beta) \subseteq N_1(\gamma)$ by construction. i.e. all vertices of $\mathbb{P}(\alpha,\beta)$ lie in $S' = S(\alpha,\beta)$ //

<u>Case</u>: $\{\alpha,\beta,\gamma\}$ is a 2-simplex in $\mathcal{C}(S)$. //

Thus $\mathbb{P}$ is a $3K$-combing. Thus $\mathcal{C}(S)$ is Gromov-hyperbolic [Gilman's criterion] //

Remarks: Bowditch uses a different criterion

["centers move continuously"



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and so deduces that the combing lines are unparametrized quasi-geodesics.

There is much more in this direction.

Most notably: Gromov-hyp spaces have a "boundary at infinity". Eg: $\partial_\infty T_3$ is a Cantor set, $\partial_\infty \mathbb{H}^2$ is a circle (for $\mathbb{H}^{n+1}$ obtain a S^n)

Rmk: If $f : X \circlearrowleft$ is a quasi isometry and X is δ-hyperbolic then the induced map $\partial_\infty f : \partial_\infty X \circlearrowleft$ is a homeomorphism. [Mostow rigidity ... etc]

Thm: [Klarreich] $\partial_\infty \mathcal{C}(S) \cong \mathcal{EL}(S)$

$\shortparallel$

Ending Laminations.

[Recall Kobayashi's proof that $\operatorname{diam}(\mathcal{C}(S)) = \infty$]