# Lecture 16:

## Fast Review

$q_t = q_t^{\alpha\beta} \in Q^1(S)$ squared surface

* $\ell_q(\delta)$ = length of geodesic rep. $\delta^*$
* $w_q(\delta)$ = max width of annuli with core curve $\simeq \delta$
* $L(q,R) = \{ \delta \in \mathcal{C}(S) \mid \ell_q(\delta) \le R \}$
* $W(q,\varepsilon) = \{ \delta \in \mathcal{C}(S) \mid w_q(\delta) \ge \varepsilon \}$
* $t_\alpha = t_{\alpha\beta}(\gamma) = \frac{1}{2}\left(K(\alpha,\beta) + K(\alpha,\gamma) - K(\beta,\gamma)\right)$

  $K(\alpha,\beta) = \log(i(\alpha,\beta))$ so need $d_S(\alpha,\beta) \ge 2$.

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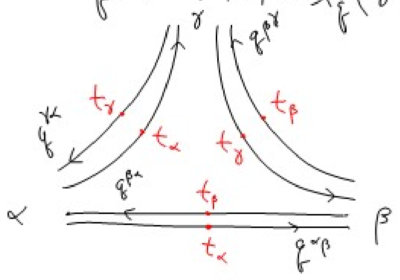
Wide annuli exist. $W(q,\varepsilon_0) \ne \emptyset$

$R = 1/\varepsilon$, $\quad \varepsilon \le \varepsilon_0$, $R_0 = 1/\varepsilon_0$.

Annulus inequality: $^{*}$ $W(q,\varepsilon) \subseteq L(q,R)$.

$(**)$ $\quad w_q(\delta) \cdot i(\delta,\gamma) \le \ell_q(\gamma)$.

$\alpha, \beta, \gamma$ all intersecting.




$$\underbrace{t_\alpha = t_{\alpha\beta}(\gamma) = t_{\alpha\gamma}(\beta)}_{\text{symmetric in } \beta,\gamma}, \; t_\beta = t_{\beta\gamma}(\alpha), \; t_\gamma = t_{\gamma\alpha}(\beta)$$

Rmk: $q_t^{\alpha\beta}$ and $q_t^{\beta\alpha}$ are identical after $90^\circ$ rotation.

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**Lemma** [Systoles have bounded diam]

$$\forall S \; \forall R \ge R_0 \; \exists K \; \forall q \in Q^1(S)$$
$$\operatorname{diam}_{\mathcal{C}(S)}\left(L(q,R)\right) \le K.$$

Picture:


Pf of Lemma:

Fix $\delta \in W(q,\varepsilon_0)$, $\gamma \in L(q,R)$

Rmk: $\delta$ exists b/c $W \ne \emptyset$.

$$\varepsilon_0 \cdot i(\delta,\gamma) \le w_q(\delta) \cdot i(\delta,\gamma) \le \ell_q(\gamma) \le R.$$

$\therefore \quad i(\delta,\gamma) \le R \cdot R_0 \le R^2$.

Hempel: $d_S(\delta,\gamma) \le 2\log_2(R^2) + 2$.

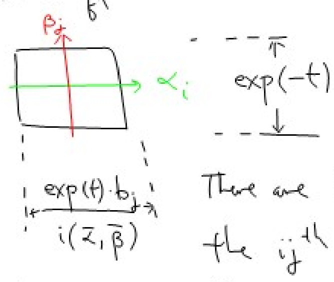
So: $\forall \gamma, \gamma' \in L(q,R) \Rightarrow d_S(\gamma,\gamma') \le 2(2\log R^2 + 2)$. //

<u>Weighted multicurves</u>: Suppose $\{\alpha_i\} \subseteq \mathcal{C}(S)$
$\{\beta_j\} \subseteq \mathcal{C}(S)$
are simplices. Let $\bar\alpha = \sum a_i \cdot \alpha_i$
$\bar\beta = \sum b_j \beta_j$
be weighted multicurves. $\left[ a_i, b_j \in \mathbb{R}_{>0} \right]$

Extend $i(\cdot,\cdot)$ <u>linearly</u> so

$$i(\bar\alpha, \bar\beta) = \sum_{i,j} a_i b_j \, i(\alpha_i, \beta_j)$$

Define $q_t^{\bar\alpha\bar\beta}$ as the union of rectangles.




There are $i(\alpha_i, \beta_j)$ of the $ij^{th}$ kind of rectangle

Glue in usual fashion to obtain $S(q_t^{\bar\alpha\bar\beta})$.

<u>Ex</u>: Check $\mathrm{Area}(S(q_t)) = 1$.

<u>Ex</u>: If $\bar\beta' = r \cdot \bar\beta \quad (r \in \mathbb{R}_{>0})$

then $q^{\bar\alpha\bar\beta}$ is <u>identical</u> to $q^{\bar\alpha\bar\beta'}$.

[i.e. only projective class of $\bar\beta$ matters.]

[Changing $\bar\alpha$ to $r \cdot \bar\alpha$ moves origin only.]

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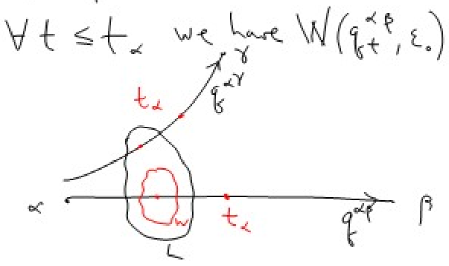
<u>Def</u>: $\ell_q(r \cdot \gamma) = r \cdot \ell_q(\gamma)$. [Very convenient. Very confusing!]

as a matter of notation.

<u>Lemma</u> [$W \leq L$ part II]

If $\alpha, \beta, \gamma$ all intersect then $\forall R \geq 3R_0$

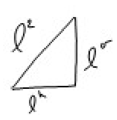
$\forall t \leq t_\alpha$ we have $W(q_t^{\alpha\beta}, \varepsilon_0) \subseteq L(q_t^{\alpha\gamma}, R)$.




<u>Def</u>:
$$\begin{cases} \alpha_t = e^{-t} \cdot \alpha \\ \beta_t = e^{t - \kappa(\alpha,\beta)} \cdot \beta \\ \gamma_t = e^{t - \kappa(\alpha,\gamma)} \cdot \gamma \end{cases}$$

Fix $\delta \in W(q_t^{\alpha\beta}, \varepsilon_0)$

WTS $\ell^2(\delta, q_t^{\alpha\gamma}) \leq 3R_0$.




<u>Exercise</u>: Suffices to bound $\ell^v, \ell^h$ of $\delta$.

Vertical $\ell^v(\delta, q_t^{\alpha\gamma}) = i(\delta, \alpha_t) = \ell^v(\delta, q_t^{\alpha\beta}) \leq \ell^2 \leq R_0$

Horizontal $\ell^h(\delta, q_t^{\alpha\gamma}) = i(\delta, \gamma_t)$

Annulus inequality $\Rightarrow$ for $q = q_t^{\alpha\beta}$

$$w_q(\delta) \cdot i(\delta, \gamma_t) \leq \ell_q(\gamma_t)$$

$$\therefore \quad \varepsilon_0 \cdot i(\delta, \gamma_t) \leq \ell_q(\gamma_t)$$

$$\therefore \quad i(\delta, \gamma_t) \leq R_0 \cdot \ell_q(\gamma_t).$$

So: It suffices to bound $\ell_q(\gamma_t)$.

Vertical: $\ell^v(\gamma_t, q) \equiv i(\alpha_t, \gamma_t) = 1$.

Horizontal: $\ell^h(\gamma_t, q) = i(\beta_t, \gamma_t)$

$$= i(\beta, \gamma) \cdot \exp(t - K(\alpha, \beta)) \cdot \exp(t - K(\alpha, \gamma))$$

$$= \frac{i(\beta, \gamma)}{i(\alpha, \beta) \cdot i(\alpha, \gamma)} \cdot e^{2t} \leq 1$$

b/c $t \leq t_\alpha = t_{\alpha\beta}(\gamma)$ //

[Think: If $t << t_{\alpha\beta}(\gamma)$ then $\exp(t - K(\alpha, \gamma))$ is small and this counterbalances the fact that $\gamma$ is long (in vert? direction)

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Next time: Define combing

$$P : \mathcal{C}^0(S) \times \mathcal{C}^0(S) \rightarrow \{\text{paths}\}$$

Do this in 2 cases.

* If $d_S(\alpha, \beta) = 1$ then $P(\alpha, \beta) = \{\alpha, \beta\}$ (the edge)

* If $d_S(\alpha, \beta) \geq 2$ then Let $S^{\alpha\beta} \subseteq S$ be the surface filled by $\alpha, \beta$. Let $q_t^{\alpha\beta} \in Q^1(S^{\alpha\beta})$ be the squared surface constructed before.

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