Lecture 15: Proving Separation

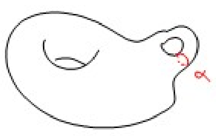
$\forall S\ \exists \eta_0\ \forall q \in Q'(S)\ \exists \alpha, \beta \in \mathcal{C}(S)$

$d_q(\alpha, \beta) \geq \eta_0$.

Recall: Any spine has length $\geq \eta_1$

Let $\eta_0 = \dfrac{\eta_1}{100 + 2n}$, $n = \text{genus}(S) + |P|$.

Also, picked $\alpha \subset S(q)$ with length arb. close to systole length.



Let $N_t = N_t(\alpha)$ the t-neighborhood of α.

$N = N_{\eta_0}$

For a contradiction suppose

Claim: $\exists$ arcs $\beta_1, \beta_2, \ldots, \beta_n$ s.t

(*) $\beta_i \subseteq N$, $\text{Length}(\beta_i) \leq 2\eta_0$, $\partial\beta_i \subseteq \alpha$

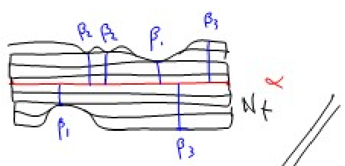
$\sigma = \alpha \cup \beta_1 \cup \beta_2 \cup \cdots \cup \beta_n$ is a spine.

If not: If there is a nontrivial region of $S \setminus N$ then we are done.

[Ie we have found separated curves α and some $\beta \subseteq S \setminus N$]

For t small N_t is an annulus. As $t \to \eta_0$ N_t gathers the topology of S. Every time N_t "bumps" find new arc β_i

Picture:

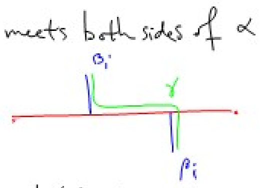


$\eta_1 \leq \text{Length}(\sigma) \leq \text{Length}(\alpha) + 2n\eta_0$

$\therefore\ 100\eta_0 \leq \text{length}(\alpha)$.

Cases: ① Some β_i meets both sides of α

Cut and paste gives curve γ s.t.



(*) $\text{Length}(\gamma) \leq \frac{1}{2}\text{length}(\alpha) + 2\eta_0$

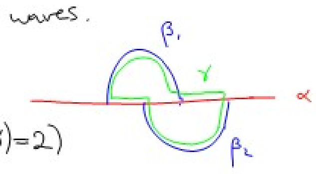
$\leq \text{Length}(\alpha) - 48\eta_0$

(**) $i(\alpha, \gamma) = 1$. So γ is ess, non peripheral

(*) $\Rightarrow$ $\text{Length}(\gamma) <$ systole length $\divideontimes$.

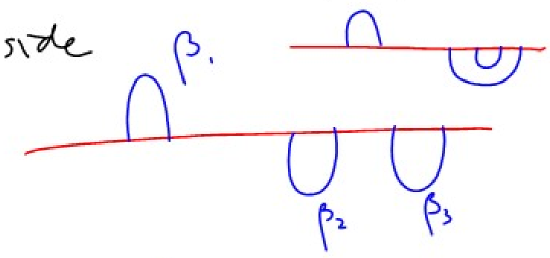
② $\exists\ \beta_1, \beta_2$ linking waves.

Check that again γ exists $(i(\alpha, \gamma) = 2)$



[Check details]

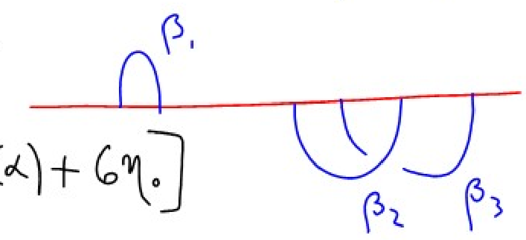
③ $\exists\, \beta_1$ on one side unlinked from β_2, β_3 on the other side



Again find γ

[This time

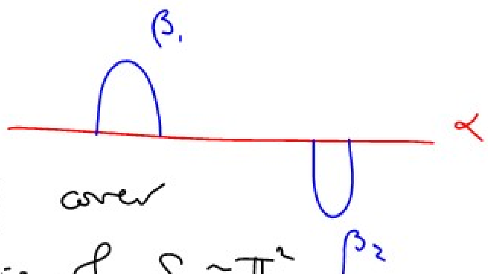
or



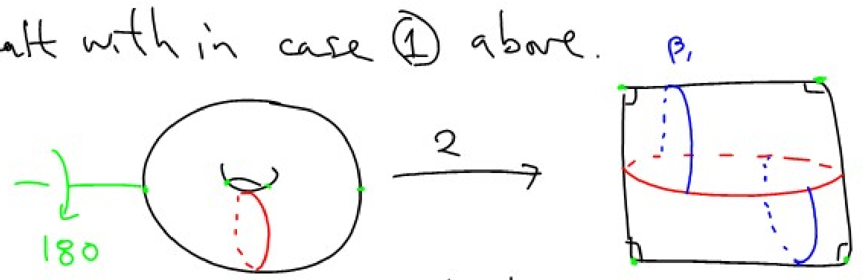
$\text{length}(\gamma) \leq \frac{2}{3}\text{length}(\alpha) + 6\eta_0$]

④ $n = 2$, no linking, α separating

Here $S \cong S_{0,4}$.



So take double branched cover and reduce to analysis of $S_1 \cong \mathbb{T}^2$ dealt with in case ① above.



This has a wide annulus, and width goes down by at most a factor of 2 under hyperelliptic.

Separation

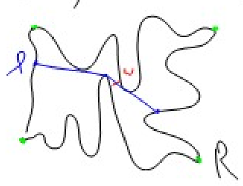
Exersize: Check cases ②, ③ carefully.

<u>Annulus inequality</u>: If A is a Riemannian annulus with width w and length ℓ

then
$$\mathrm{Area}(A) \geq w \cdot \ell.$$

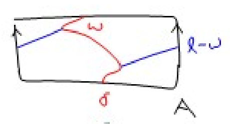
[ℓ = length of systole
= inf length of core curves]

<u>Pf</u>: Rectangle inequality
$$\mathrm{Area}(R) \geq w \cdot \ell$$



Let A be any annulus, let $\delta \subseteq A$, $\partial_\pm \delta \in \partial_\pm A$ s.t. length of δ is as close as needed to w.

Cut A along δ to find
$$\mathrm{Area}(A) \geq w \cdot (\ell - w)$$
[via Rectangle inequality for $A \setminus \delta$.]



Let A_n be the n-fold cover of A

<u>Easy</u>: $\mathrm{Area}(A_n) = n \cdot \mathrm{Area}(A)$

<u>Exercise</u>: $w(A_n) = w(A)$ [Easy]

<u>Exercise</u>: $\ell(A_n) = n \cdot \ell(A)$

$$n \cdot \mathrm{Area}(A) = \mathrm{Area}(A_n) \geq w(A_n) \cdot (\ell(A_n) - w(A_n))$$
$$\geq w \cdot (n\ell - w)$$

$$\therefore \quad \mathrm{Area}(A) \geq w \cdot \ell - \frac{w^2}{n} \xrightarrow[n \to \infty]{} w \cdot \ell. \quad //$$

<u>Corollary</u>: $\forall \gamma \in \mathcal{C}(S)$, $\forall q \in Q^1(S)$
$$w_q(\gamma) \cdot \ell_q(\gamma) \leq 1.$$

<u>Pf</u>: A is an (almost) optimal annulus for γ

so $\underbrace{w_q(\gamma) = w_q(A)}_{\text{optimal}}$, $\underbrace{\ell_q(\gamma) \leq \ell_q(A)}_{\forall A}$

<u>So</u>: $w_q(\gamma) \cdot \ell_q(\gamma) \leq w_q(A) \cdot \ell_q(A) \leq \mathrm{Area}(A)$
$$\leq \mathrm{Area}(S) = 1. \quad //$$

<u>Wide Curves</u> <u>Define</u>: $W(q, \varepsilon) =$
$$\{\gamma \in \mathcal{C}(S) \mid w_q(\gamma) \geq \varepsilon\}.$$

Wide annuli $\Rightarrow$ if $\varepsilon \leq \varepsilon_0$ then $W(q, \varepsilon) \neq \emptyset$.

Set $R_0 = 1/\varepsilon_0$, $R = 1/\varepsilon$.

<u>Thus</u>: By corollary of Annulus inequality $\boxed{W(q, \varepsilon) \subseteq L(q, R)}$

<u>Observe</u>: If $\delta, \gamma \in \mathcal{C}(S)$, $q \in Q^1(S)$
$$w_q(\delta) \cdot i(\delta, \gamma) \leq \ell_q(\gamma). \quad \text{[Exercise]}$$