Lecture 14 Wide Annuli.

$S(q)$ a half translation surface.

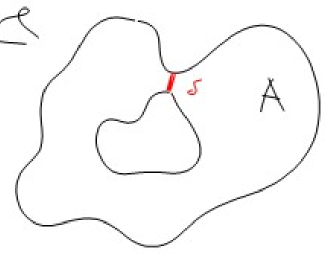
$A \subseteq S$ is an embedded annulus. $\partial A = \bigcup_{\varepsilon=+,-} \partial_\varepsilon A$

$q|A$ is the restricted metric.

$w_q(A) =$ width of A with metric $q|A$

$$= \inf \left\{ \ell_{q|A}(\delta) \;\middle|\; \begin{array}{l} \delta \text{ is an arc in } A \\ \text{connecting } \partial_\pm A \end{array} \right\}$$

Picture

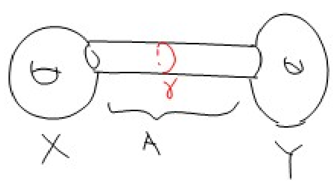


Define, for $\gamma \in \mathcal{C}(S)$

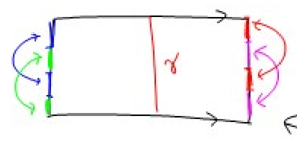
$$w_q(\gamma) = \sup \left\{ w_q(A) \;\middle|\; \begin{array}{l} A \subseteq S \text{ annulus} \\ \text{with core} \\ \simeq \gamma \end{array} \right\}$$

Question: Do optimal annuli exist? Are they nice?

Example:



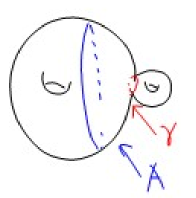
Example



Wide flat annuli about γ

Here Area(X) = Area$(Y) = 0$.

Example



Wide but not flat.

Lemma [Wide Annulus Exists] $\forall S\ \exists \varepsilon_0 > 0$ $\forall q \in Q^1(S)$ [$S(q)$ is a half translation surface of area 1] $\exists \gamma \in \mathcal{C}(S)$ s.t. $w_q(\gamma) \geq \varepsilon_0$.

I.e. "Every half-translation surface has a curve of definite width."

Masur-Minsky Proof: "Geometric limit argument in the Deligne-Mumford compactification of moduli space."

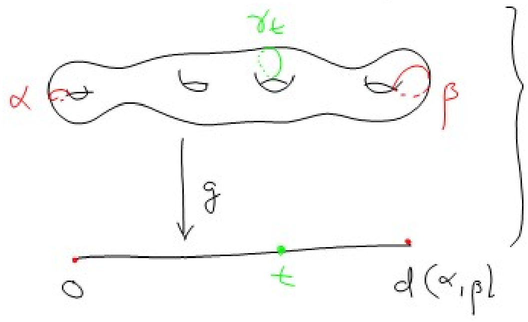
Bowditch's Proof (is constructive)

Separation Lemma: $\forall S\ \exists \eta_0 > 0\ \forall q \in Q^1(S)$
$\exists\, \alpha, \beta \in \mathcal{C}(S)$ s.t. $d_q(\alpha,\beta) \geq \eta_0$

Separation $\Rightarrow$ wide annuli: Given α, β that are η_0-separated. Let
$g : S(q) \longrightarrow [0, d_q(\alpha,\beta)]$ be defined by

$$g(x) = \min\{ d_q(\alpha, x), d_q(\alpha,\beta) \}$$

Exercise: g is distance non-increasing.



Via very small pert. we may assume g smooth generic.

Let $m = \xi(S) + 1$
$= 3\,\text{genus} + |P| - 3 + 1.$

where $P \subseteq \overline{S}(q)$ are the punctures of $S(q)$.
$[\overline{S}(q)$ is the metric completion$]$

Note that $g^{-1}(t)$ separates α from β so contains an ess. nonperipheral simple closed curve $\gamma_t \subseteq g^{-1}(t)$.

Let $\gamma^i = \gamma_{\left(i \cdot \frac{d(\alpha,\beta)}{m}\right)} \subseteq g^{-1}\left(i \cdot \frac{d(\alpha,\beta)}{m}\right)$

Since $m > \xi(S)$ $\exists\, i \neq j$ with $\gamma^i \simeq \gamma^j$.

Hence: The annulus cobounded by γ^i, γ^j has width $\geq |j - i| \frac{d(\alpha,\beta)}{m} \geq \frac{\eta_0}{m}$. // Sep $\Rightarrow$ wide.

Pf of Separation:

[Question: Is there an easier proof of Sep Lemma? Or of Wide annulus Lemma?]

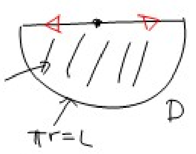
Def: A trivial region $D \subseteq S$ is an embedded disk with at most one point of P in the interior.

Let $f(x) = \frac{1}{2\pi}x^2$. Any such D satisfies

$\text{Area}(D) \leq f(\text{Length}(\partial D))$. ["Isop. inequality" for S]

Example:

$A = \frac{\pi}{2}r^2$

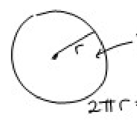


D

$\pi r = L$

$\frac{\pi}{2}r^2 \leq \frac{1}{2\pi}(\pi r)^2$

or



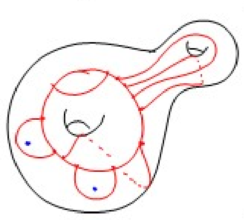
$\pi r^2 = A$

$2\pi r = L$

$\pi r^2 \leq \frac{1}{2\pi}(2\pi r)^2 = 2\pi r^2$.

Exercize?: Prove this for $S(q)$.

Def: A graph $\sigma \subseteq S$ is a spine if $S \setminus \sigma$ is a union of trivial regions



This spine is not minimal [i.e. it's too long!]

If σ is minimal then $|S \setminus \sigma| = \begin{cases} 1 & \text{if } P = \emptyset \\ |P| & \text{if } P \neq \emptyset \end{cases}$

The largest (area) piece of $S \setminus \sigma$ has area $\geq \frac{1}{\max\{1, |P|\}}$ So: $\text{Length}(\sigma) \geq \eta_1 \equiv f^{-1}\left(\frac{1}{\max}\right)$.

So. Any spine has length $\geq \eta_1$.

Let $n = \text{genus}(S) + |P|$.

Define: $\boxed{\eta_0 = \frac{\eta_1}{100 + 2n}}$

Def: $\text{systole}(S(q)) = \inf\{\ell_q(\alpha) \mid \alpha \in \mathcal{C}(S)\}$

Let $\alpha \subseteq S(q)$ be a simple closed curve (ess. non per)

s.t $\text{Length}(\alpha) < \text{systole}(S(q)) + \eta_0/10^{10^{10}}$

Let N_t = t-neighborhood of α

$N = N_{\eta_0}$. [Ponder: How does N_t "evolve" as t goes from 0 to η_0] ← Next Time