Lecture 12

K. Strebel "Quadratic differentials"
A. Zorich "Flat surfaces"

<u>Names</u>: Squared surfaces, Singular flat metrics, half-translation surfaces, quadratic differentials [rational billiard tables] ... [Interval Exchange Transform]

<u>Notation</u>: $S(q)$ is the surface S equipped with the singular flat metric q.

<u>Last time</u>: Define $S(q_t^{\alpha\beta})$ as union of rectangles $\{R_x \mid x \in \alpha \cap \beta\}$ and rectangles where all identical, in $\mathbb{R}^2$.
[Area$(q) = 1$.]

<u>Generalize</u>: ① Equip S with an atlas of charts in $\mathbb{R}^2$ s.t. all overlap maps are translations $(z \longmapsto z+a)$ or half translations $(z \longmapsto -z+a)$; except at a finite number of points which are modelled on branched covers of $\mathbb{R}^2/_{z\sim -z}$, branched over the origin. [and possibly remove the cone point]

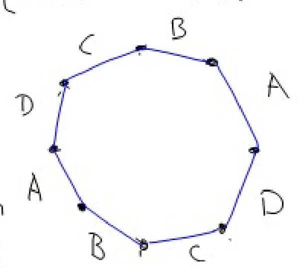
[Note all cone angles are multiples of π].

② Let $\{P_i\}$ be a collection of polygons in $\mathbb{R}^2$ equipped with a complete collection of <u>edge pairings</u>

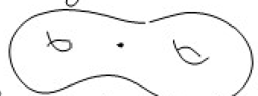
[identify parallel sides of P_i, P_j via translation or half translation.]

Glue to obtain <u>orientable</u> $S(q)$ [q is metric induced from $\mathbb{R}^2$] and remove fin. many points. [Must remove all cone pts of angle π.]

<u>Octagon</u>:



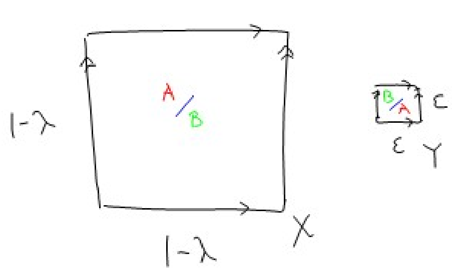
Take the regular octagon of area 1 and glue opposite sides by translation.

Topologically get 
S_2 with conepoint of angle 6π.

[Hint: an n-gon may be triangulated with $n-2$ triangles.]

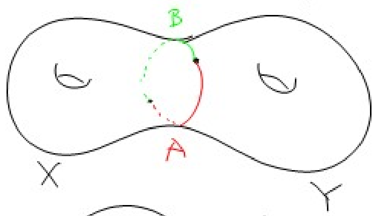
Example ② "Thick and thin decomposition"

An "unbalanced surface"



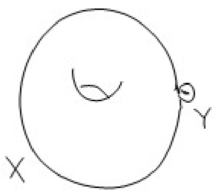
Suppose are square tori of area $(1-\lambda)^2$ and ε^2 so that $(1-\lambda)^2+\varepsilon^2=1$.

Cut X, Y along parallel slits of length δ. Arrange $1 \gg \lambda, \varepsilon \gg \delta > 0$. Glue sides of slits as indicated. Topologically we get.



and find 2 cone points of angle 4π.

Metrically we see



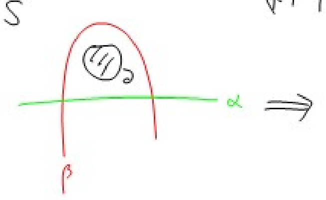
Notice that it is easy to find [Exercise!] curves $\alpha, \beta \subseteq S$ s.t. the lengths of $\alpha, \beta \leq 2$ But $i(\alpha, \beta) \geq \left(\frac{1}{\varepsilon}\right)^2$ [perhaps even larger? an additional $1/\delta$]

Thus: Unlike in $\mathbb{H}^2$-geometry moderate length does not ensure moderate complexity.
So need information addition to length if we wish to control topological complexity.

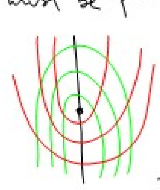
Geodesic rep'ives in half translation surfaces.

Fix $S(q)$.
(*) All cone angles $\in \mathbb{N} \cdot \pi \setminus \{0\}$
(*) punctures not boundary components
(*) All cone points with angle π must be punctures.

S



$\Longrightarrow$



vertical / horiz foliations of $S(q)$ near π-point.

Notation: If $\alpha \in \mathcal{C}_0(S)$ Let α^* be the geodesic rep of α in $S(q)$.

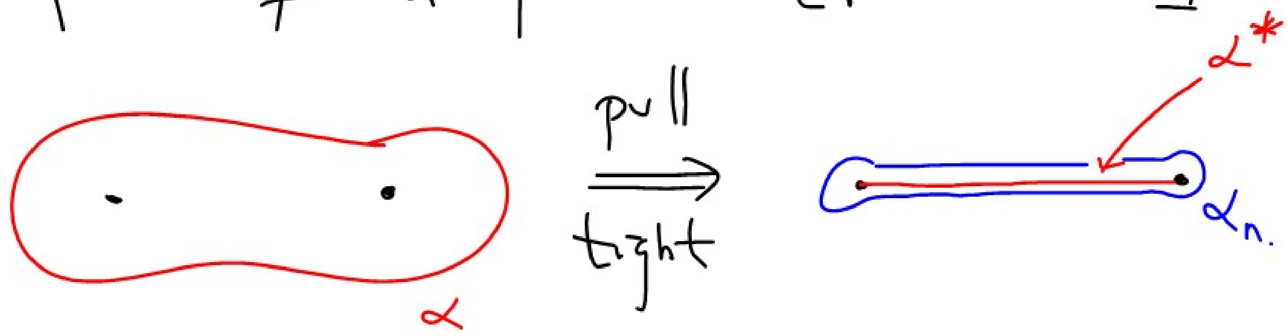
Difficult exercise: Prove existence and uniqueness‡ of α^*. [cf Strebel].

In fact: ① α^* is a union of $\mathbb{E}^2$ segments $\{\sigma\}$ with $\partial\sigma \subset \{\text{cone pts}\}$ $\forall \sigma$.

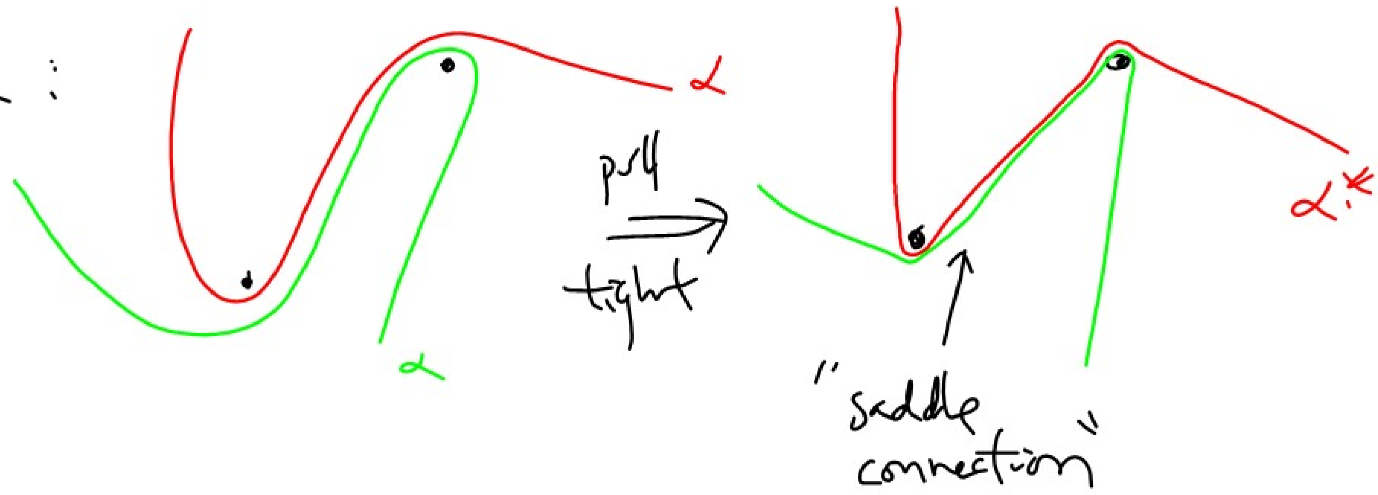
② At cone pts of angle $\geq 2\pi$ α^* has $\geq \pi$ angle on both sides.

③ α^* need not be embedded but there are nearly geodesic rep'ives α_n s.t. $\alpha_n \to \alpha^*$ [Hausdorff] and $\text{leng}(\alpha_n) \to \text{leng}(\alpha^*)$

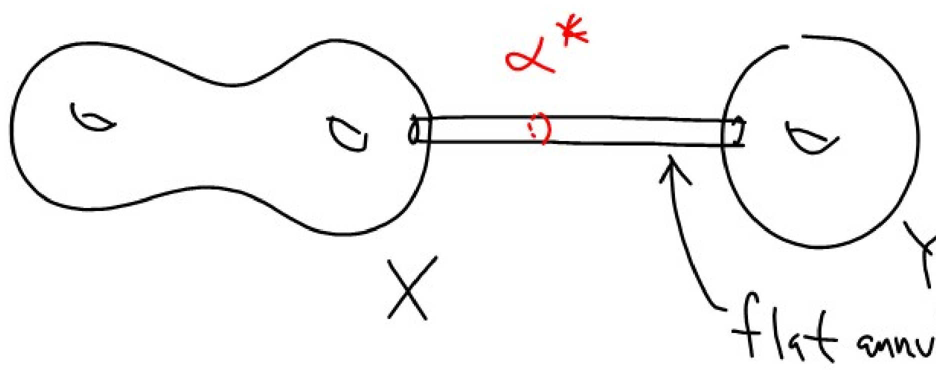
Examples: If α cuts off a disk with a pair of π-points [plus etc...]



Example:



Finally: ‡ refers to the possibility of flat annuli



Geodesic rep's of α foliate A.