## Lecture 11

**Prop (E):** [Masur Minsky]
If 3-quasi geodesics are $R$-stable then geod. triangles are $R$-slim.

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This is the final implication

$$\text{slim} \Rightarrow \text{combings} \overset{A}{\Rightarrow} \text{chords} \overset{B}{\Rightarrow} \text{subquad} \overset{C}{\Rightarrow} \text{linear} \overset{D}{\Rightarrow} \text{stable} \overset{E}{\Rightarrow} \text{slim}.$$

A,B : Gilman's papers
C : Bowditch [Gromov]
D : " "
E : Masur Minsky, I.

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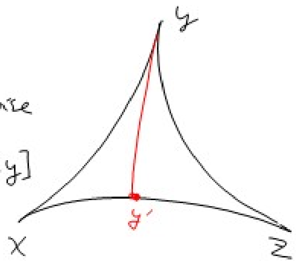
**Pf of E:** Pick $x, y, z \in X$. $T = T_{xyz}$

Let $y'$ be a closest point of $[x,z]$ to $y$.




**Exercise:** The piecewise geodesic $[x,y'] \cup [y',y]$ is a 3-quasi-geod.

So $[x,y'] \subseteq N_R([x,y])$. Similarly $[y',z] \subseteq N_R([y,z])$. So $T$ is $R$-slim. //

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**Exercise:** If $X \underset{q.i.}{\simeq} Y$ and $X$ is Gromov hyperbolic then so is $Y$.

**Exercise:** If $\mathbb{Z}^2 \underset{q.i.emb}{\hookrightarrow} X$ then $X$ is not Gromov hyperbolic.

**Harder:** Suppose $G$ is a group (fin. gen) and $\mathbb{Z}^2 < G$. Then $G$ is not Gromov hyperbolic.

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**Goal:** [Masur Minsky] $\mathcal{C}(S)$ is Gromov hyp.

Prove this following Bowditch:
"Intersection numbers and the hyperbolicity of the $\mathcal{C}\mathcal{C}$."

**Basic idea:** The systole map. applied to geodesics in Teichmuller space, gives a slim combing of $\mathcal{C}(S)$.

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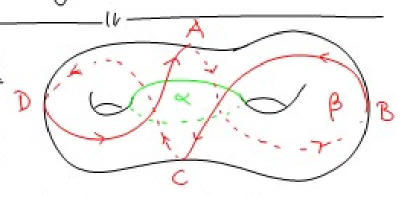
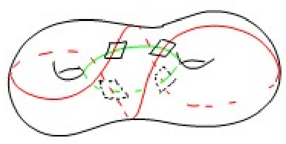
**Squared Surfaces**



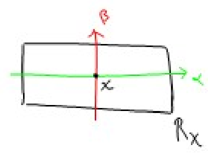

**Def:** $\alpha, \beta$ fill $S$ if $\forall \gamma$ either $i(\gamma,\alpha) > 0$ or $i(\gamma,\beta) > 0$.

If we pull $\alpha, \beta$ <u>tight</u> (i.e. $|\alpha \cap \beta| = i(\alpha,\beta)$) then the graph $\alpha \cup \beta \subset S$ has no bigons.

For any $t \in \mathbb{R}$ we build a <u>singular flat surface</u> $q_t^{\alpha\beta} \underset{\text{homeo}}{\cong} S$ as follows.



For every $x \in \alpha \cap \beta$ let $R_x$ be the rectangle




of height $\exp(-t) = e^{-t}$ and width $\dfrac{\exp(t)}{i(\alpha,\beta)}$

<u>Notation</u>: $K(\alpha,\beta) = \log(i(\alpha,\beta))$

So width is $\exp(t - K(\alpha,\beta))$.

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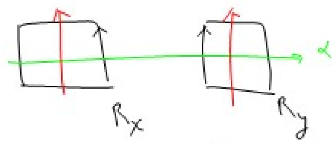
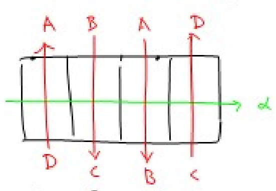



Glue $R_x$ to $R_y$ along a $\left\{\begin{matrix} \text{vertical} \\ \text{horiz} \end{matrix}\right\}$ side

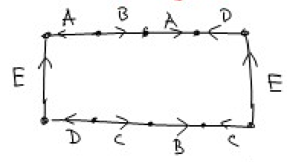
if $x$ is connected to $y$ by an arc of $\left\{\begin{matrix} \alpha \setminus \beta \\ \beta \setminus \alpha \end{matrix}\right\}$.

[<u>Rmk</u>: Perhaps a suitable introduction, and refs, can be found in early papers of Rafi.]

<u>Example</u>:

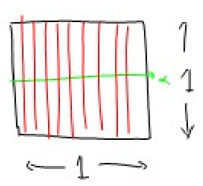



i.e.

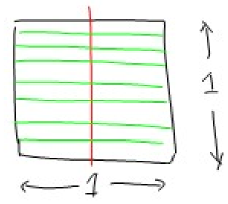



$t = 0$

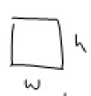
(i)



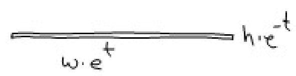

$t = K(\alpha,\beta)$




(ii)




General rectangle.




(iii) $\alpha$ has an annular neighborhood of width 1.

$\beta$ has an ann. neigh. of width 1.

$\boxed{\text{Qbert.}}$

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<u>Singularities</u>: the angle at a vertex of $q_t^{\alpha\beta}$ is a $\mathbb{N}$-multiple of $\pi$.