Lecture 10

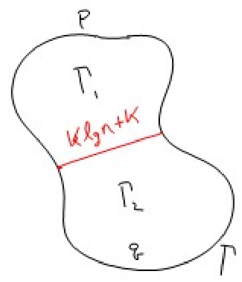
Proposition B: If X has K-short chords then X satisfies a subquadratic isoper. inequality. $[\, f(n) \le n^{1.9} \,]$

Pf: Suppose $K \geq 1$. Pick any $N > 0$ so that $\dfrac{K \cdot \log_2(N) + K}{N} \le 1/16$.

Suppose that $\Gamma \in \Omega(X) = \{\text{loops}\}$.

Induct on $L(\Gamma) = \text{length}(\Gamma) = n$.

Base case: If $n \le N$ then $\text{Area}_N(\Gamma) = 1 \le n^{1.9}$



$p + q = n$. $p, q \in [\frac{n}{4}, \frac{3n}{4}]$

So if $p \le q$, $p \in [\frac{n}{4}, \frac{n}{2}]$

$$L(\Gamma_1) \le p + \underline{K \lg n + K} \le \frac{3n}{4} + \frac{n}{16} < n$$

Also $L(\Gamma_2) < n$

$\Rightarrow A(\Gamma_1), A(\Gamma_2)$ satisfy induction hypothesis.

$$\text{(A1)} \quad A(\Gamma) \le A(\Gamma_1) + A(\Gamma_2)$$
$$\le (p + K\lg n + K)^{1.9} + (q + K \lg n + K)^{1.9}$$
$$\le n^{1.9}\left[\left(\frac{p}{n} + \frac{K\lg n + K}{n}\right)^{1.9} + \left(\frac{q}{n} + \frac{K \lg n + K}{n}\right)^{1.9}\right]$$
$$\le n^{1.9}\left[\left(\frac{p}{n} + \frac{1}{16}\right)^{1.9} + \left(\frac{q}{n} + \frac{1}{16}\right)^{1.9}\right]$$
$$\le n^{1.9}\left[\left(\frac{1}{4} + \frac{1}{16}\right)^{1.9} + \left(\frac{3}{4} + \frac{1}{16}\right)^{1.9}\right]$$
$$\le n^{1.9}$$

// cf Gilman's paper.

Proposition C [Gromov]

If X satisfies a subquad. isop. ineq then X " " linear " " .

[In fact, we have shown that short chords $\Rightarrow$ strictly subquadratic.
Eg better than $f(n) = \dfrac{n^2}{\log n}$]

Ref: See Bowditch's paper "A short proof that a subquadratic...."

Also discusses axiom (A2); contains references.

The proof first controls chords then iteratively shows subquad $\Rightarrow$ strictly subquad $\Rightarrow$ linear.

$\varepsilon \cdot n^2$	$\dfrac{n^2}{\lg n}$	$n^{1.9}$	n
very small c	$o(n^2)$	$o(n^{2-c})$	$O(n)$
	subquad	strictly subquad.	linear

Quasi-Geodesics:

Suppose $G : [p,q] \longrightarrow X$ is a path
$\qquad\quad\ \ \cap\!|$
$\qquad\quad\ \ \mathbb{Z}$

and a Q-quasi-isom. embedding.

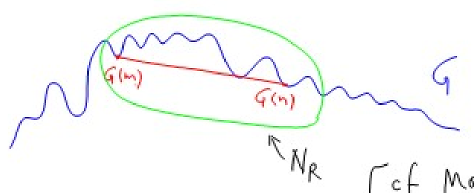
Call G a Q-quasi-geodesic.

Exercise: Classify Q-quasi geod in T_3.

Super impossible: " " " " in $\mathbb{R}^2$.

Def: We say $\underset{\text{a path}}{G}$ is R-stable if $\forall\, m,n \in [p,q]$

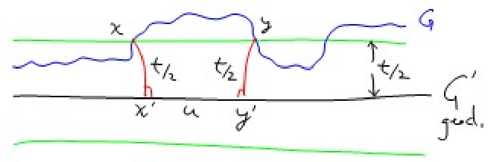
$$G|[m,n] \subseteq N_R\big(\,[G(m), G(n)]\,\big)$$



[cf Morse stability]

Prop ⓓ $\forall K, Q\ \exists R$ so if X has K-linear isop. ineq and G is a Q-quasi-geodesic then G is R stable.

Pf:



G is quasi. G' is geod. with same endpoints

$M = 2K^3 Q'$, $Q' = Q+1$, $t = M+1$.

let $x, y \in G$ where G exits then reenters $N_{t/2}(G')$. [Rmk if x does not exist, we are done.]

Let x', y' be any closest points of G' to x, y. [inf realized as G' compact.].

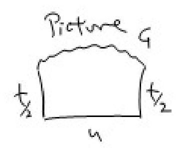
$$\Gamma = [x,x'] \cup [x',y'] \cup [y',y] \cup \big(G|_{y,x}\big)$$

is a rectangle. let $u = L([x',y'])$

$L = L(\Gamma)$, $A = A(\Gamma)$

Quasi: $L \leq u + t + Q(u+t) + Q$
$\qquad\qquad \leq Q'(u+t) + Q'$

Picture



Linear: $A \leq KL + K$

$(A2)_K$: $\frac{t}{2}(u-t) \leq K^2 A$.

Calculate:

$$\begin{aligned} t(u-t) &\leq 2K^2 A \leq 2K^2(KL+K) \\ &\leq 2K^3 L + 2K^3 \leq 2K^3\big(Q'(u+t) + Q'\big) + 2K^3 \\ &\leq 2K^3 Q' u + 2K^3 Q' t + 2K^3 Q' + 2K^3 \\ &\leq Mu + Mt + 2M \end{aligned}$$

$$tu - Mu \leq t^2 + Mt + 2M$$

$$u \leq t^2 + Mt + 2M.$$

This bounds the length of $G|_{x,y}$ So $R = \frac{t}{2} + \frac{Q}{2}(u+t) + \frac{Q}{2}$ suffices. //

Exercise: Find X $\forall K \exists \Gamma$ a loop s.t. $A_K(\Gamma)$ is undefined.