Lecture 9

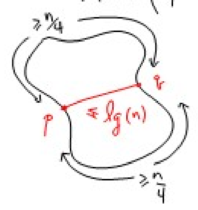
Theorem: If X admits a δ-slim combing then X is Gromov hyperbolic.

Pf: Will use ideas of Gilman, Bowditch, and many others.

(I) Chords

Suppose $\Gamma \subset X$ is a loop. $n = \text{length}(\Gamma)$. A pair of points $p, q \in \Gamma$ are *almost antipodal* if

$$\text{len}(\Gamma|_{p,q}),\ \text{len}(\Gamma|_{q,p}) \geq \frac{n}{4}.$$



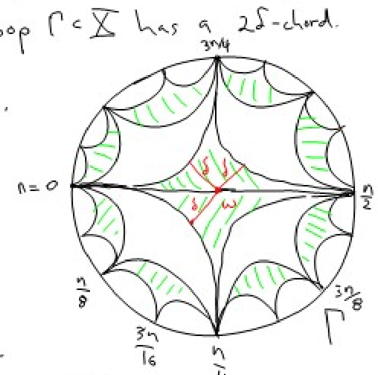
Say (p,q) is a *K-chord* if

(*) $d_X(p,q) \leq_K \log_2(n)$

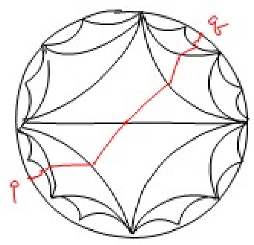
(**) p, q almost antipodal.

Proposition (A): If X has a δ-combing then every loop $\Gamma \subset X$ has a 2δ-chord.

Pf: Fix Γ, a loop.



Since $R(0,\frac{n}{2})$ is connected there is a point $w \in R(0,\frac{n}{2})$ that is in the δ-neighborhood of both $R(\frac{n}{2},\frac{3n}{4})$ and $R(\frac{3n}{4},n)$. WLOG w is in a δ-neighborhood of $R(0,\frac{n}{4})$.



Now use δ-slim R-Δ's to find almost antipodal p, q s.t.

$$d_X(p,q) \leq 2\left(\delta \cdot \lg_0\left(\tfrac{n}{2}\right) + \delta\right) \leq 2\delta \log_2(n).$$

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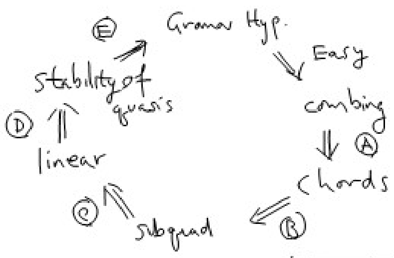
Outline of Pf of thm.

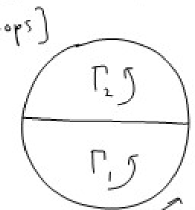


Area and isoperimetric inequalities

Properties of "area": $A : \Omega \longrightarrow \mathbb{R}_{\geq 0}$, $\Omega = \{\text{loops}\}$

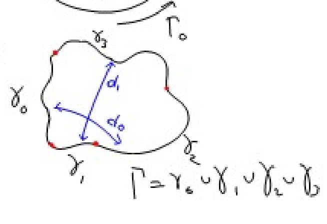
(A1) Theta inequality

$$A(\Gamma_0) \leq A(\Gamma_1) + A(\Gamma_2)$$



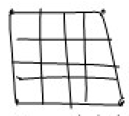
(A2) Rectangle inequality

$$d_0 \cdot d_1 \leq A(\Gamma)$$



Example: Suppose X is a graph.
Let $\mathbb{D}$ be the unit disk. Let $\mathbb{P}$ be a K-cellulation of $\mathbb{D}$

* $\mathbb{P}'$ is a graph
* $\partial\mathbb{D} \subset \mathbb{P}'$
* all components of $\mathbb{D} \setminus \mathbb{P}'$ are open disks (faces)
* the perimeter of every face has edge length $\leq K$.



4-cellulation.

Def: $\mathrm{Area}_K : \Omega_X \to \mathbb{N}$

$$\mathrm{Area}_K(\Gamma) = \min \left\{ |\mathbb{P}| = \#\text{faces} \;\middle|\; \begin{array}{l} \mathbb{P} \text{ a K-cell.} \\ \exists \text{ a simplicial map} \\ f : \mathbb{P}' \to X \text{ s.t. } f(\partial\mathbb{D}) = \Gamma \\ \text{[degree one]} \end{array} \right\}$$

Exercise: Show $(A2)_K$

$$d_0 \cdot d_1 \leq K^2 \mathrm{Area}_K(\Gamma)$$

Def: (X, A) has isoperimetric inequality $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ if $\forall \Gamma \subset \Omega$ (loops)

$$A(\Gamma) \leq f(\mathrm{Len}(\Gamma)).$$

Linear isoper. inequality if

$$A(\Gamma) \leq_Q \mathrm{Len}(\Gamma)$$

Exercises: $\mathbb{Z}^2$ has quadratic inequality.
[Show $\mathbb{R}^2$ has quad inequality.]

Exercise: T_3 has a linear inequality.

[This is all related to Dehn functions and solving the word problem in groups]

Prop Ⓑ: $\forall K \, \exists N$ If X has K-short chords then Area_N has a subquad isoper. inequality. [In fact, $f(n) = n^{1.9}$.]

Pf: Next time. //