<u>Last time</u>: $\mathcal{F} \not\cong T_3, \mathbb{H}^2$. Suggestion made to use "volume growth". Perhaps this will not work? Consider $(\mathbb{N}, d(i,j)=1\ \forall i\neq j)$

This is quasi-isom to a point.

<u>Def</u>: $\mathcal{C}_k(S)$ is the graph with vertices $\mathcal{C}^0(S)$ and $\{\alpha,\beta\}$ is an edge if $i(\alpha,\beta) \leq k$.

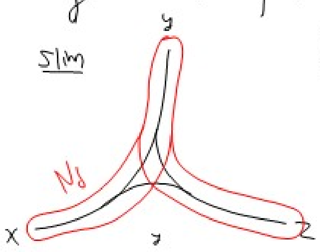
<u>Exercise</u>: $\mathcal{C}_k(S) \underset{qi}{\cong} \mathcal{C}(S)$.

[<u>Question</u>: What about using $i(\alpha,\beta)=k$ for edges? <u>Ans</u>: Perhaps still qi to $\mathcal{C}(S)$ if k has correct parity?? Perhaps not? Urk?]

<u>Hyperbolic spaces after Gromov</u>.

Suppose X is a geodesic metric space. $\forall x,y \in X$ let $[x,y]$ be any geodesic connecting x to y. $\forall x,y,z$ Let $T = T_{xyz} = [x,y] \cup [y,z] \cup [z,x]$ be a geodesic triangle.

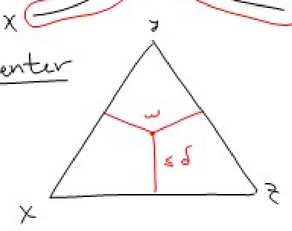
<u>Slim</u>



(Rips)

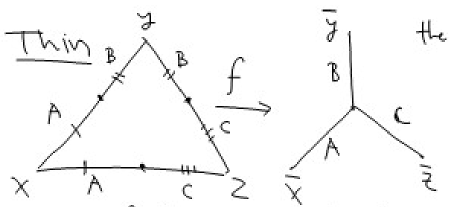
<u>Def</u>: T is <u>δ-slim</u> if $[x,z] \subseteq N_\delta([x,y]\cup[y,z])$

<u>Center</u>



<u>Def</u>: T has a <u>δ-center</u> if $\exists w \in X$ s.t $w \in N_\delta([p,q])$ for $p \neq q$, $p,q \in \{x,y,z\}$

<u>Thin</u>



the associated tripod

$A = \frac{1}{2}(\overline{xy}+\overline{xz}-\overline{yz})$

$B = \frac{1}{2}(\overline{yx}+\overline{yz}-\overline{xz})$

$C = \frac{1}{2}(\overline{zx}+\overline{zy}-\overline{xy})$

$f: T \longrightarrow T(A,B,C)$

<u>Def</u>: T is <u>δ thin</u> if $\forall t$ in the <u>tripod</u> $\operatorname{diam}_X(f^{-1}(t)) \leq \delta$. [The tripod $T(A,B,C)$ is the metric space shown.]

<u>Def</u>: X is <u>Gromov hyperbolic</u> (δ-hyperbolic) if $\exists\delta$ $\forall$ triangles T, T is δ-slim.

<u>Exercise</u>: If X is δ-hyperbolic then $\exists \delta', \delta''$ s.t all triangles have δ'-centers, are δ''-thin.

<u>Exercise</u>: Actually prove that all three conditions are equivalent (making later δ's larger as needed.)

Easy
<u>Exercise</u>: Fixing Gromov hyperbolic X The optimal δ's satisfy $\delta_{center} \leq \delta_{slim} \leq \delta_{thin}$.

Exercise: Trees are 0-hyperbolic.

Exercise: $\mathbb{R}^n$ ($\mathbb{Z}^n$) is δ-hyp iff $n = 0, 1$.

Exercise: $\mathbb{H}^n$ is δ-hyperbolic.

Perhaps: $\delta_{center} = \log(\sqrt{3})$

$\delta_{slim} = \log(1+\sqrt{2})$

$\delta_{thin} = \log\left(\frac{3+\sqrt{5}}{2}\right)$.

Question: If $\delta_{center} = \delta_{thin}$ is X a tree?

[Guess: No: there will be some counter example...]

Gilman's Hyperbolicity Criterion: Suppose X is a graph with all edges of length 1.

A path in X means an edge path

A loop in X " " closed path.

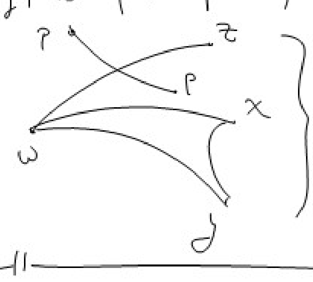
A combing of X is a map

$$R : X^{(0)} \times X^{(0)} \longrightarrow \{\text{paths}\}$$

$$(x, y) \longrightarrow R(x,y) \text{ a path from } x \text{ to } y.$$

[if $x = y$ $R(x,y)$ is path of length zero]

Picture:



Def: A combing R is δ-slim if

(i) R-triangles are δ-slim.

$\forall x,y,z \quad R(x,z) \subseteq N_\delta\big(R(x,y) \cup R(y,z)\big)$

(ii) if $d_X(x,y) = 1$ then $\operatorname{len}\big(R(x,y)\big) \leq \delta$.

Observe: If X is δ-hyperbolic then the geodesic combing is δ-slim.

Theorem: If X admits a δ-slim combing then X is Gromov hyperbolic.

This is hard!