(I) Coarse Geometry

$r, s, A \in \mathbb{R}_{\geq 0}$, $A \geq 1$.

$r \leq_A s$ if $r \leq As + A$.

$r =_A s$ if $r \leq_A s$ and $s \leq_A r$.

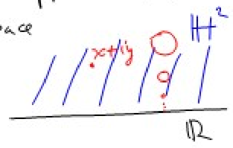
[Almost transitive!]

Suppose X, Y are geodesic metric spaces

[a geodesic $[x, y]$ from x to y has $\forall z \in [x, y]$ $d(x, z) + d(z, y) = d(x, y)$

So $\forall x, y \in X$ there is a geodesic $[x, y]$ connecting them.

Exercise: Let $\mathbb{H}^2$ be the upper half plane model of hyperbolic space



The element of length is $ds_{\mathbb{H}} = ds_{\mathbb{E}}/y$.

Check that $\mathbb{H}^2$ is a geodesic metric space.

In fact it is always enough to suppose that X, Y are graphs with all edge lengths $\equiv 1$

Def: A function $f: X \to Y$ is a Q-quasi-isometric embedding if $\forall x, x'$

$$d_X(x, x') =_Q d_Y(f(x), f(x'))$$

[$y, y' = f(x), f(x')$.

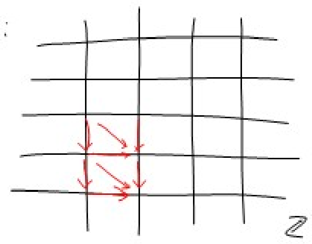
if $d_Y(y, y') \leq_Q d_X(x, x')$ then say f is coarsely Lipschitz]

If, in addition, $Y \subseteq N_Q(f(X))$

$= Q$-neighborhood
$= \{y \in Y \mid \exists x \in X$ and $d_Y(y, f(x)) \leq Q\}$

then call f a Q-quasi-isometry.

Example:



$\mathbb{Z}^2 \subseteq \mathbb{R}^2$

Exercise: the function $f: \mathbb{R}^2 \to \mathbb{Z}^2$ is a quasi-isometry. [Which metrics?]

[Perhaps work in L^1 metric]

Exercise: $(\mathbb{R}^2, L^p) \underset{q.i.}{\simeq} (\mathbb{R}^2, L^2)$. [allow $p = \infty$]

Exercise: The relation quasi-isometry is an equivalence relation.

[symmetry is nontrivial!]

<u>Exercise</u>: $\mathbb{Z}^n \simeq \mathbb{Z}^m$ iff $m=n$

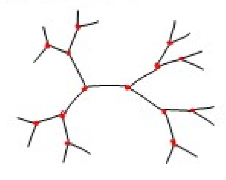
<u>Gromov's Program</u>: Classify groups (generally, spaces) up to quasi-isometry

<u>Rmk</u>: If X has finite diameter then $X \simeq \{pt\}$. Thus all finite groups are qi to each other [i.e. have qi Cayley graphs]

<u>Exercise</u>: If G is a fin. gen group and S, T are gen. sets then $\Gamma_S \underset{qi}{\simeq} \Gamma_T$ [Cayley graphs for S, T]

NB: If G is not fin. gen. then we may still discuss Γ_S (now locally ∞) but the above Exercise no longer holds.

<u>Exercise</u>: Recall T_k = k-regular tree



Determine the qi classes in $\{T_k \mid k = 0, 1, 2, 3, \ldots = \omega\}$

[Murmurings: Not very many of them...]

<u>Corollary</u>: Do this for $\mathbb{F}_n$ = free group on n gens.

<u>Exercise</u>: $\mathbb{H}^n \underset{qi}{\simeq} \mathbb{H}^m$ iff $n=m$.

<u>Exercise</u>: Do qi classification for

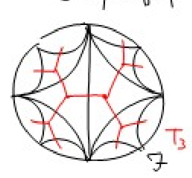
$$\{\mathbb{H}^n, \mathbb{E}^n, T_n \mid n \in \mathbb{N}\}$$

<u>Question</u>: Suppose that $X \underset{qi\ emb}{\overset{qi\ emb.}{\rightleftarrows}} Y$

Does this imply $X \underset{qi}{\simeq} Y$??

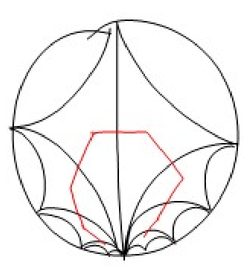
<u>Fact</u> <u>Bowditch</u>: $\mathcal{F} \underset{qi}{\simeq}$?

$\mathcal{F} \not\simeq \mathbb{H}^2$: The obvious map fails b/c $d_{\mathcal{F}}(0,\infty) = 1$ but $d_{\mathbb{H}^2}(0,\infty) = \infty$ ($0, \infty \notin \mathbb{H}^2$?)



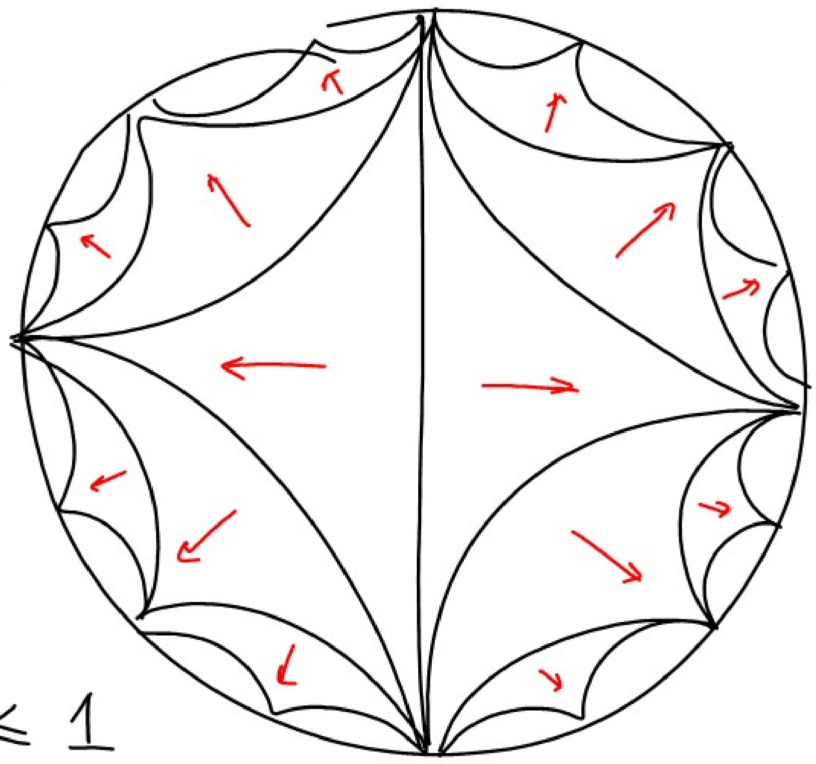
In fact the obvious map is not a map.

$\mathcal{F} \neq T_3$: Perhaps consider Volumes of balls of radius n.



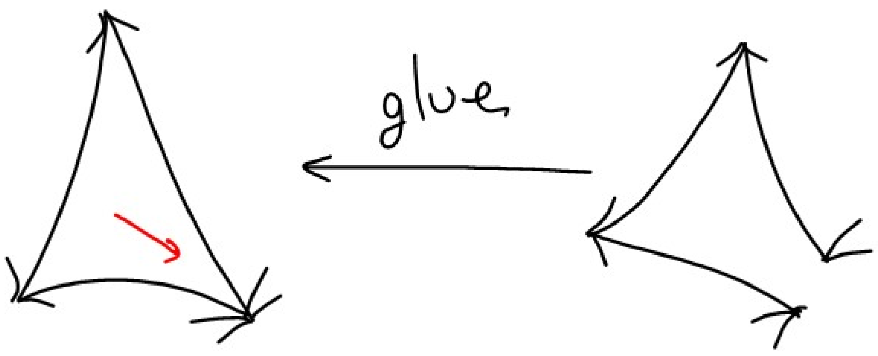
we see that the obvious inclusion fails b/c The red infinite geodesic shown is contained in a finite diam. set.

Pf by picture



Every triangle has a red arrow, every vertex is pointed at by ≤ 1 arrow.

Now break open all triangles at the vertex



After breaking vertices we find T_ω. The map that glues vertices is a quasi-isometry

[Check this!]