Lecture 6

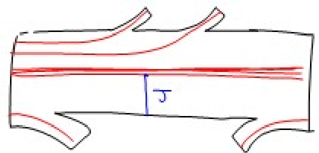
Theorem: $\operatorname{diam}(\mathcal{C}(S)) = \infty$.

In the middle of showing: If ω is Keane then every non-boundary leaf of $\mu = \mu_\omega$ is dense.

Pf: Suppose not. Let λ be the non dense leaf. Let $C = \overline{\lambda}$. Claim: $\exists\ \partial$-leaf disjoint from C. Pf [Last time.]

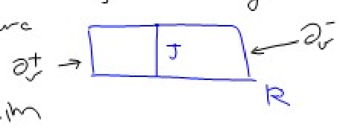
Given the claim, let J be a maximal vertical arc so that $\operatorname{int}(J) \cap C = \emptyset$, $\partial_+ J \in C$, $\partial_- J \in \partial N(\omega)$

Picture:



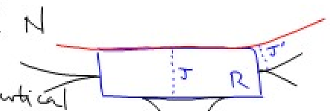
Take J to the the shortest such vertical arc [There are finitely many boundary leaves]

Let R be a maximal rectangle containing J as a vertical arc



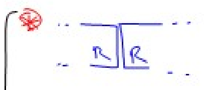
Identical to the claim

⊛ $\partial_v^+ R \cap \partial_v^- R = \emptyset$, Thus $\partial_v R$ must meet cusps of N

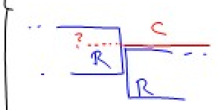


Thus there is a vertical arc J', shorter than J and satisfying the hypotheses on J $\divideontimes$ //

[⊛



} If $\partial_v^+ = \partial_v^-$ then find a smooth cycle $\divideontimes$.



} If the vertical sides meet but are not equal then $C \cap \operatorname{int}(J) \neq \emptyset$ $\divideontimes$]

This proves the theorem. //

Exercise: For any measure $\omega \in W(\tau)$ and for any $\lambda \in \mu_\omega$ the closure of λ is homeomorphic to either ① a smooth cycle or ② a tie neighborhood of a track $\sigma < \tau$.

[If you include maximal annuli of smooth cycles then we may decompose μ_ω as a finite union of 'minimal components' such]

Thm: Suppose $\omega \in W(\tau)$ is Keane. Then there is a sequence of splittings of τ to $\tau' < \tau$ s.t

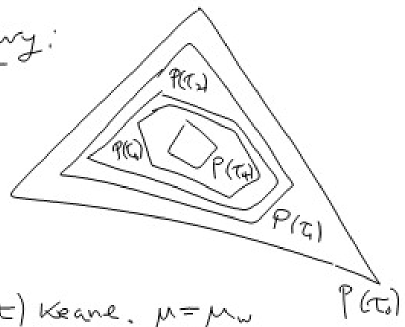
(∗) τ' is maximal

(∗∗) $\omega \in W(\tau')$

(∗∗∗) $P(\tau') \subset \operatorname{int}(P(\tau))$.

Corollary: $\exists$ sequence $\tau = \tau_0 > \tau_1 > \tau_2 > \cdots$ s.t. τ_i maximal, $P(\tau_i) \subset \operatorname{int}(P(\tau_{i-1}))$, and $\omega \in \bigcap_{i=0}^{\infty} P(\tau_i)$. Corollary $\operatorname{diam}(\mathcal{C}_S) = \infty$.

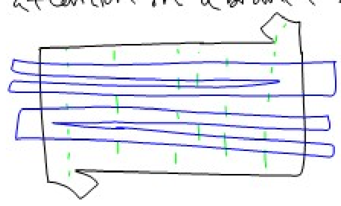
Picture for corollary:



Pf of Theorem

Given $w \in W(\tau)$ Keane. $\mu = \mu_w$

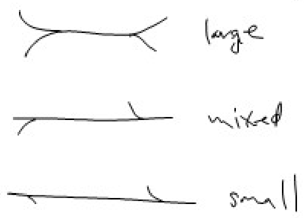
Fix attention on a branch b of τ.



split along all subarcs of singular leaves until the cusps all cross b, as shown.

Do this for all small branches b. to obtain τ'.

[Recall



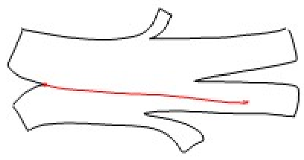
]

So: All small branches of τ' cross all small branches of $\tau \Rightarrow$

$P(\tau') \subset \operatorname{int} P(\tau)$ //

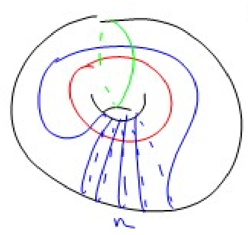
Check: $\cdot\, \tau'$ maximal
$\cdot\, w \in W(\tau')$

[Recall: To split means to choose a compact subarc of a singular leaf



NB $\mathcal{C}(S)$ is locally infinite so it is hard to "see" large distance.

Eg:



The $\frac{n}{1}$ slope "looks" far from the $\frac{0}{1}$ slope but $d_{\mathcal{C}}(\frac{0}{1}, \frac{n}{1}) \leq 2$, in fact.

Exercise: Draw a pair of curves in $S_{g,n}$ at distance 4. [Easy for $S_{1,1}$, $S_{0,4}$

Harder for $S_{0,5}$

Harder for $S_{2,0}$.

Exercise: Give, constructively, an example of $w \in W(\tau)$ with the Keane property.

[Secret info: If $f : S \circlearrowleft$ is pseudo-Anosov and $\alpha \in \mathcal{C}^0(S)$ is a curve then the orbit $\{f^k(\alpha)\}_{k \in \mathbb{Z}}$ is a quasi geodesic $\Rightarrow$ infinite diameter.

<u>New Topic</u> <u>Coarse Geometry</u>.

<u>Basics</u>: $A \geq 1$, $r, s \in \mathbb{R}_{\geq 0}$ write

$$r \leq_A s \quad \text{if} \quad r \leq A \cdot s + A$$

$$r =_A s \quad \text{if} \quad r \leq_A s, \; s \leq_A r$$

Suppose that X, Y are geodesic metric spaces [In fact, suffices to consider graphs where all edge lengths are <u>one</u>.]

<u>More next time.</u>