Lecture 5:

Draw a surface. Draw a train track.
[Not $\mathbb{T}^2$] [And not $S_{0,4}$]

*) Check that all regions have neg. index.

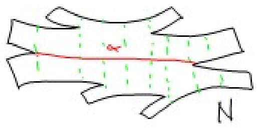
**) No smooth boundary.

__Exercise__: Find $\omega : B(\tau) \to \mathbb{R}_{\geq 0}$ so that ω satisfies the switch conditions __and__ $\omega(b) > 0 \;\forall b \in B(\tau)$.

WTS $\operatorname{diam}(\mathcal{C}(S)) = \infty$

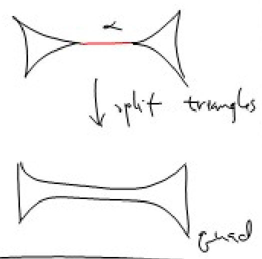
__Exercise__: Suppose τ maximal, nice.
Suppose $\alpha < \tau$ is a carried arc.
$\tau_\alpha = \tau$ split along α

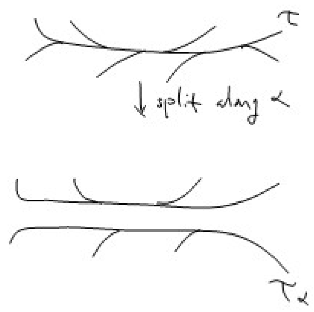
$\dim(W(\tau_\alpha)) < \dim(W(\tau))$



Notice that τ_α is not maximal.

Eg





Notice that τ carries (up to tie preserving isotopy) only countably many arcs. [Exercise]

__Def__: $\omega \in W(\tau) \setminus \bigcup_{\alpha < \tau} W(\tau_\alpha)$ iff ω has the __Keane property__ wrt τ.

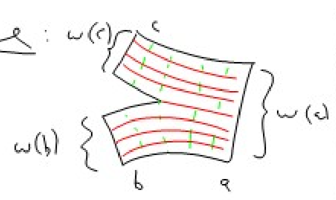
[Remark: It is possible, using pseudo-Anosov maps to find (constructively) ω with Keane property.

__Exercise__: If τ is standard track in S_1 ($\mathbb{T}^2$) then ω is Keane iff μ_ω is a __foliation__ of irrational slope.

II __Foliations__ If $\omega \in W(\tau)$ define

$N(\omega) = \bigcup_{b \in B(\tau)} R_b(\omega) \big/$ glue vertical sides. where $R_b(\omega)$ has height is $\omega(b)$ and its width is __one__.

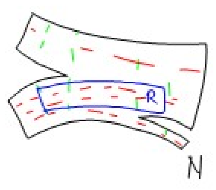
__Picture__:



Let $\mu_\omega = \mu$ be the horizontal foliation by leaves.

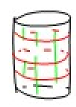
__Theorem__: If ω is Keane then every non boundary leaf of μ is dense in $N(\omega)$.

III) Rectangles A rectangle $R \subseteq N(w)$ is a map of $R \subset \mathbb{E}^2$ (coord rectangle) s.t. vertical/horizontal arcs in R are sent to ties/leaves and the map restricted to $\mathrm{int}(R)$ is an embedding.

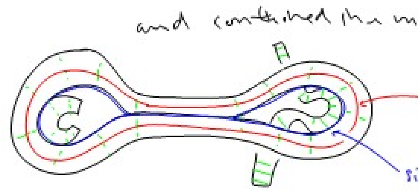


and vertical/horiz measures pull back to dx/dy.

Similarly could define annuli:



Def: An immersion $S^1 \looparrowright N(w)$ is a (smooth) singular cycle if the image is horizontal. (and contained in smooth leaves) and contained in a union of boundary and singular leaves.



IV) Lemmas: w is Keane, $\mu = \mu_w$

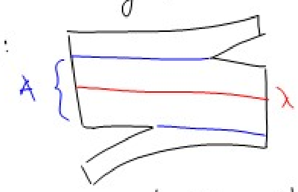
Lemma: μ has no singular cycles.

Pf: μ has no compact singular leaves. //

No loop lemma: μ has no smooth cycles.

Pf: Suppose $\lambda \subset \mu$ is a smooth cycle. So there is a small neighborhood of λ disjoint from ∂N. Let A be a maximal area annulus containing λ.

Picture:



Since A is maximal $\partial_\pm A$ either meets itself or meets a cusp (as shown above)

If $\partial_+ A = \partial_- A$ then $S = \mathbb{T}^2$ and μ has rational slope $\divideontimes$ [Exercise]

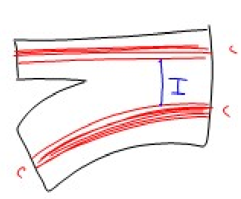
If $\partial_+ A$ meets a cusp the $\partial_+ A$ is a singular cycle $\divideontimes$. // No loops.

Dense Thm: Non-boundary leaves are dense.

Pf: Suppose λ is not a boundary leaf. Let $C = \overline{\lambda}$ and suppose $C \neq \mu$.

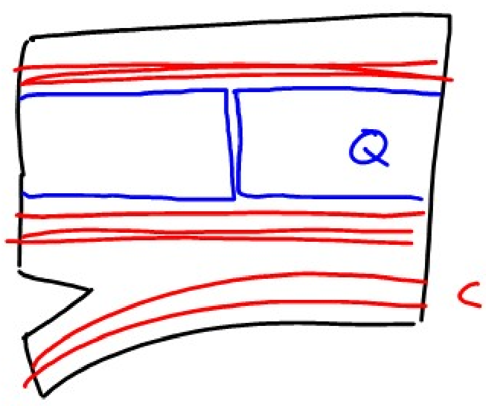
Claim: $\exists$ a boundary leaf not contained in C.

Pf: Since $C \neq \mu$ $\exists$ some vertical arc I in N s.t. (*) $I \cap C = \partial I$



Let Q be any maximal area rectangle containing I. Let $\partial_v^I Q$ be the vertical sides

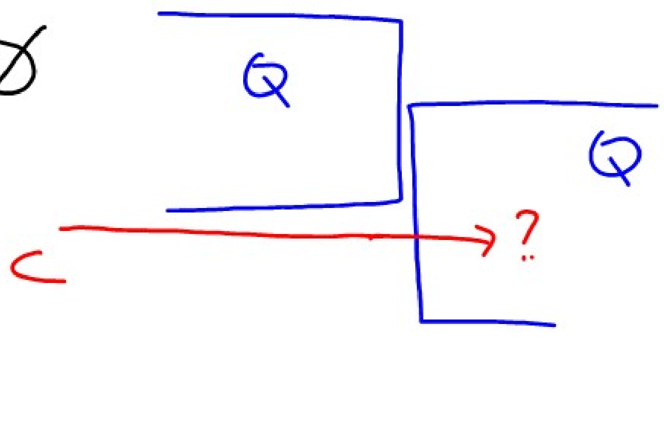
Cases: ① If $\partial_v^+ Q = \partial_v^- Q$



Then there is an annulus

$\Rightarrow \exists$ smooth cycle $\divideontimes$.

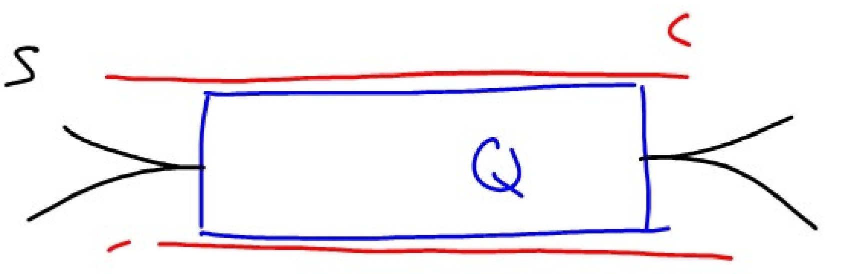
② if $\partial_v^+ Q \cap \partial_v^- Q \neq \emptyset$



Then $c \cap \mathrm{int}(F) \neq \emptyset$

contradiction.

③ Thus $\partial_v Q$ meets cusps



and this proves the claim. //

Finish next time.