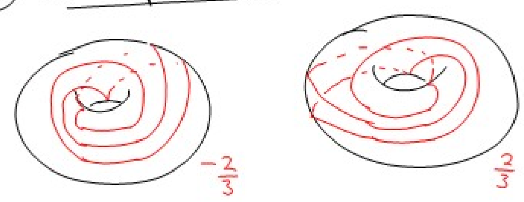
Lecture 3

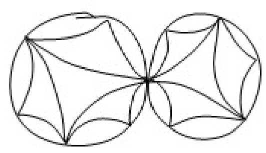
(I) Cleanup last time.



Exercise: 5 properties characterizing $\mathcal{F}$.

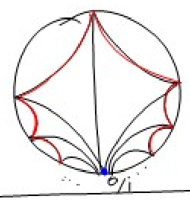
Add vi) Every vertex link is connected

Eg rules out



Def: If K is a simplicial complex and $\sigma \in K$

Then $\operatorname{link}(\sigma) = \{\tau \in K \mid \sigma \cap \tau = \emptyset,\ \sigma \cup \tau \in K\}$



$\operatorname{lk}(\frac{0}{1}) \cong \mathbb{Z}$.

(II) Goal for next two weeks.

Thm: $\operatorname{diam}(\mathcal{C}(S)) = \infty$.

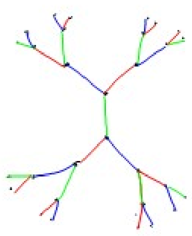
Exercise: $\operatorname{diam}(\mathcal{F}) = \infty$.

Trees: T_k be the k-regular tree

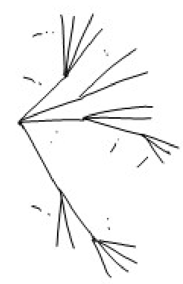
k 0   1   2

T_k ·   —— $\mathbb{Z}$

$k=3$

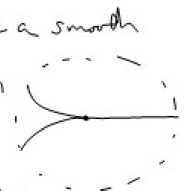


$k=\infty$



(III) Train tracks

A train track $\tau \subset S$ is a smooth graph locally modelled on



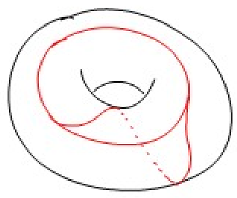
Terminology [Thurston]

The vertices are called switches

" edges " branches

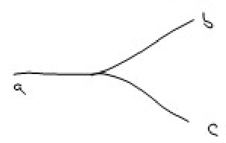
$\mathcal{S}(\tau) = \{\text{switches}\}$, $\mathcal{B}(\tau) = \{\text{branches}\}$

Tracks first defined as "branched 1-submanifolds" [Williams]



Weights [Transverse measures]

$W(\tau) = \{ w : \mathcal{B} \to \mathbb{R}_{\geq 0} \mid w(a) = w(b) + w(c) \text{ at every switch} \}$



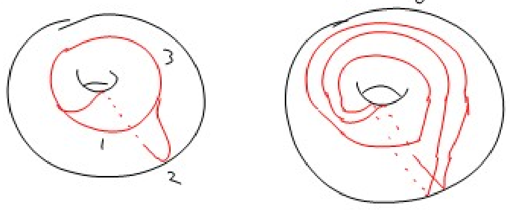
$w(a) = w(b) + w(c)$.

NB: As in the example: the switch equations may not linearly independent.

Def: $P(\tau) = \mathbb{P}W(\tau) = \dfrac{W(\tau) - \{0\}}{\mathbb{R}_{>0}}$

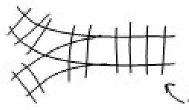
Typically P is a compact, convex, polyhedron.

Notice that if $w \in W(\tau)$ is an integral point then there is a corresponding multicurve α_w

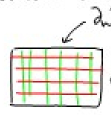


Exercise: Compute $W_{\mathbb{Z}}(\tau) \subseteq \{\text{slopes}\}$ with τ as shown.

(IV) Tie neighborhoods:

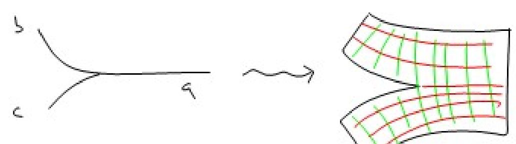


Define $N = N(\tau) \subseteq S$ by taking a collection $\{R_b \mid b \in B(\tau)\}$ of rectangles



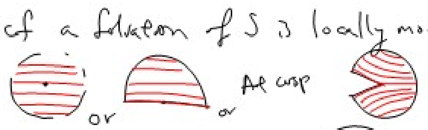
green ties are vertical
red leaves are horizontal

Now glue $\partial_v R_a, \partial_v R_b, \partial_v R_c$ according to the switches



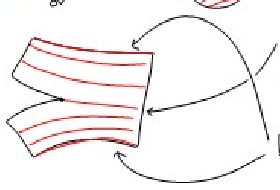
The leaves/ties fit together to from a pair of foliations [horiz/vertical]

[Def a foliation of S is locally modelled on



]

Names:

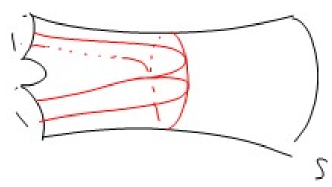


All other leaves are called smooth leaves.

Ref: Penner–Harer "Combinatorics of Train Tracks"
Mosher 350-page monograph.

(V) Conditions on tracks

(1) No smooth boundaries Every component of ∂N meets a cusp.



(2) Negative index: Every region of $S \setminus N$ has negative index

A region T is the closure of a component of $S \setminus N$. $\text{index}(T) = \chi(T) - \frac{1}{2}\#\text{cusps of } T$

T 

index 1 $\quad \frac{1}{2}$ $\quad 0$ $\quad -\frac{1}{2}$ $\quad -\frac{1}{2}$

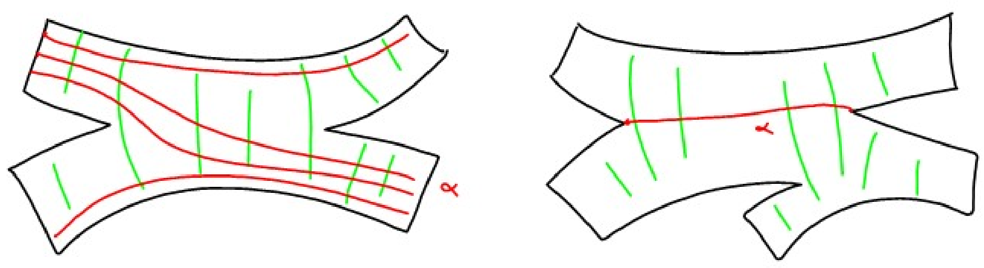
VI Carried arcs and curves.

Def: If $\alpha \subset N$, an arc or curve, is

(*) transverse to ties

(**) $\partial\alpha \subset \{\text{cusps}\}$

then α is carried by τ; $\alpha < \tau$



Def: If $\alpha < \tau$ is a curve then

$$w_\alpha(b) = |\alpha \cap t|, \; t \text{ any tie in } R_b$$

and $w_\alpha \in W(\tau)$

Exercise: If $\alpha < \tau$ (a curve) then α is essential and non peripheral. [Neg. index]

Harder Ex: If $\alpha, \beta < \tau$ and $\alpha \simeq \beta$ then there is an isotopy of α to β preserving the tie structure of N.

Remark: by the exercises there is a map.

$$P_{\mathbb{Z}}(\tau) \longrightarrow \mathcal{C}_0(S).$$