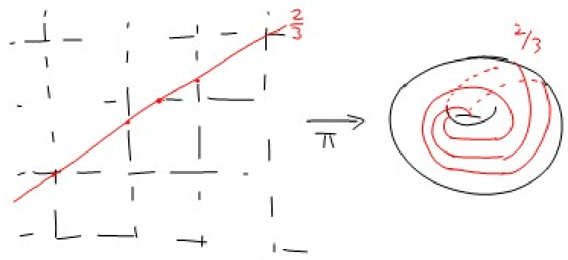
Ⓥ Consider $\mathbb{T}^2 = S_1$. Note

$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$.

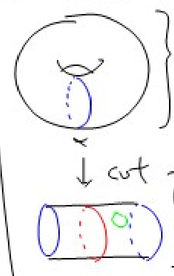
Exercise: ① If $L \subset \mathbb{R}^2$ is a line of rational slope then $\pi(L) \in \mathcal{C}^0(\mathbb{T}^2)$

② if $L, L'$ have same rational slope then $\pi(L) \simeq \pi(L')$

③ [Harder] Every $\alpha \in \mathcal{C}^0(\mathbb{T}^2)$ is isotopic to $\pi(L)$, $L$ rational. Say the slope of $\alpha$ is the slope of $L$.




NB: With previous definition $\mathcal{C}(\mathbb{T}^2)$ has no edges.




if $i(\alpha,\beta) = 0$ then $\beta \simeq \alpha$.

only contains the core curve and the trivial curve.

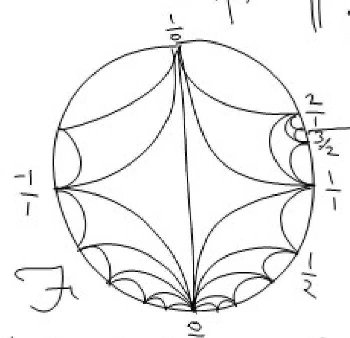
Fix this:

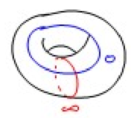
$\Delta \subseteq \mathcal{C}^0(\mathbb{T}^2)$ is a simplex iff $\forall \alpha, \beta \in \Delta$, $i(\alpha,\beta) \equiv 1$.

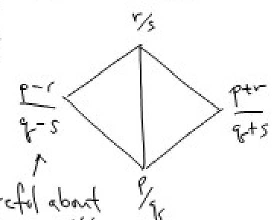
Def: $i(\alpha,\beta) = 1$ iff $\alpha, \beta$ are Farey neighbors

Exercise: If $\alpha, \beta$ have slopes $\frac{p}{q}, \frac{r}{s}$

Then $i(\alpha,\beta) = \left\| \begin{matrix} p & r \\ q & s \end{matrix} \right\|$




$\mathcal{F}$ = the Farey tesselation

Careful about signs!!!

Farey addition

Rmk: Farey addition only "works" for Farey neighbors.

Exercise: Prove that this picture is correct.

① $|\Delta| \leq 3$. (at most triangles)

② Every vertex meets $\infty$-many edges

③ " edge " exactly 2 triangles.

④ Every edge separates $\mathcal{F}$.

⑤ $\mathcal{F}$ is connected.

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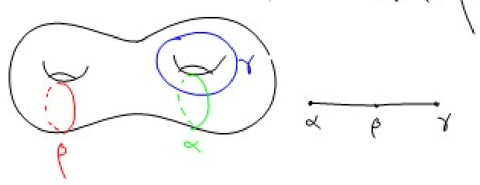
Exercise: $\underbrace{\mathcal{C}(S_1) \cong \mathcal{C}(S_{1,1})}_{i(\alpha,\beta)=1} \cong \underbrace{\mathcal{C}(S_{0,4})}_{i(\alpha,\beta)=2} \cong \mathcal{F}$

Exercise: Draw the slope $1/2$. [Draw $3/2$ if too easy]

## VI Metric on $\mathcal{C}(S)$. Suppose $\alpha, \beta \in \mathcal{C}^0(S)$

<u>Define</u> $d_S(\alpha, \beta) =$

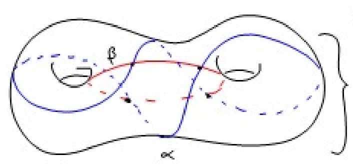
$$\min \left\{ |P| \;\middle|\; P \text{ an edge path from } \alpha \text{ to } \beta \right\}$$




<u>Review</u>: $d_S(\alpha,\beta) \le 1 \iff \alpha, \beta$ disjoint $(i(\alpha,\beta)=0)$

$d_S(\alpha,\beta) \le 2 \iff \exists\, \gamma$ disjoint from both.

$\iff \alpha, \beta$ do <u>not</u> fill $S$.




$d_S(\alpha,\beta) = 3$

[check this!]

<u>Lemma</u> [Hempel] [Bowditch, MM, Lickorish, Ivanov, Dehn... various versions]

If $i(\alpha,\beta) \neq 0$ then

$$d_S(\alpha,\beta) \le 2 \cdot \log_2(i(\alpha,\beta)) + 2 \quad //$$

<u>Rmk</u>: Sharper version for $\mathcal{F} = \mathcal{C}(\mathbb{T}^2)$

if $\alpha, \beta \in \mathcal{F}^0$ and $\alpha \neq \beta$

$$d_{\mathcal{F}}(\alpha,\beta) \le \log_2(i(\alpha,\beta)) + 1$$

[Pf is easier than pf of Hempel's Lemma]

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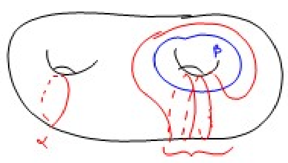
Neither of these inequalities can be "reversed".




$\beta_n$.

<u>Note</u>: $d_S(\beta, \beta_n) = 2$ (if $n \neq 0$)

but $i(\beta, \beta_n) = n$.

<u>Good Question</u>: Can Hempel's inequality be improved when $i(\alpha,\beta)$ is large?

<u>Rmk</u>: Due to existence of pseudo-Anosov maps, the log must appear.

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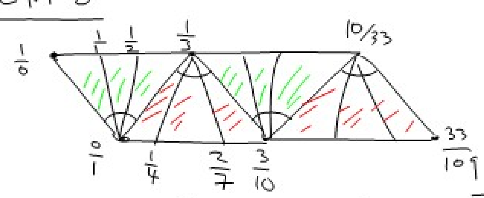
<u>Corollary</u>: $\mathcal{C}(S)$ is connected.

<u>Example in $\mathcal{F}$</u>




[Note $S_{n+1} = 3S_n + S_{n-1}$]

<u>Notice</u>: $$\frac{33}{109} = \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3}}}} = [0,3,3,3,3]$$

<u>Call</u>: $\frac{0}{1}, \frac{1}{3}, \frac{3}{10}, \frac{10}{33}$ the <u>pivots</u>. <u>Thus</u> $d_{\mathcal{F}}\left(\frac{1}{0}, \frac{33}{109}\right) \le 5$

① Prove this is an equality ② Show $d_{\mathcal{F}}\left(\frac{1}{0}, \frac{p}{q}\right) =$ length of cont. frac. of $p/q$.