Hyperbolicity of the curve complex

Goal: Prove a theorem of Masur and Minsky:

Thm The curve complex is Gromov hyperbolic.

Outline: Curve complex,
notions of hyperbolicity
singular flat structures

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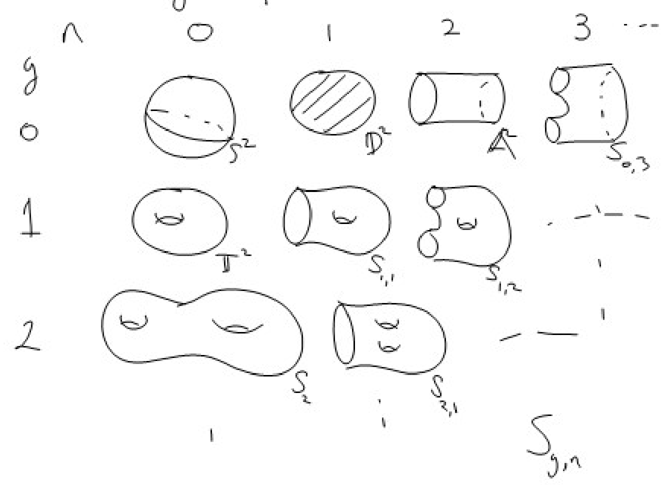
website for course

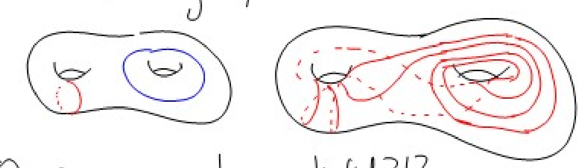
(I) Surfaces

Surfaces: Connected, compact, orientable surfaces are determined by their genus and the number of boundary components.



(II) Curves

Curves: A curve $\alpha \subset S$ is a embedding of $S^1 \hookrightarrow S$



Q: curves can be compliated?!?

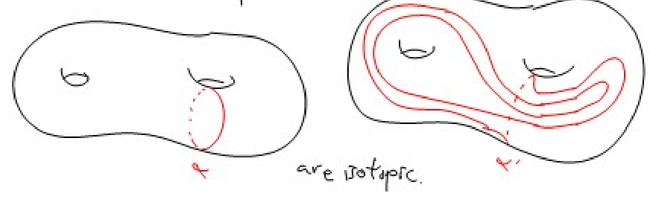
Isotopy: An isotopy is a continuous map $F: S \times I \to S$ s.t. if we define $F_t : S \to S$ by $F_t(x) = F(x,t)$ then F_t is a homeomorphism $\forall t \in I = [0,1]$

Def: $\alpha, \beta \subset S$ are isotopic curves if there is an isotopy F s.t.

* $F_0 = Id_S$
** $F_1(\alpha) = \beta$

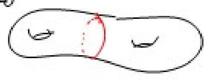
[Exercise: Show $\simeq$ is an equivalence relation]

[Write $\alpha \simeq \beta$.]

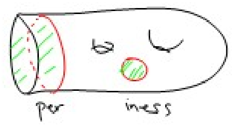


are isotopic.

<u>Def</u>: α is <u>separating</u> if $S \setminus \alpha$ not connected.



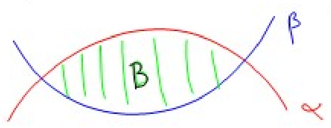
<u>Def</u>: If α separates and $S \setminus \alpha$ has a $\left\{ \begin{array}{l} \text{disk} \\ \text{annulus} \end{array} \right\}$ component then α is $\left\{ \begin{array}{l} \text{inessential} \\ \text{peripheral} \end{array} \right\}$



<u>Def</u>: $[\alpha] = \{ \beta \text{ a curve} \mid \beta \simeq \alpha \}$

<u>Def</u>: $\mathcal{C}^0(S) = \left\{ [\alpha] \;\middle|\; \begin{array}{l} \alpha \text{ essential and} \\ \text{non peripheral} \end{array} \right\}$

(III) <u>Bigons</u>: If $B \subset S \setminus (\alpha \cup \beta)$ is a disk and $\overline{B}$ meets α, β each in a single arc then B is a <u>bigon</u>.



Thus can reduce $|\alpha \cap \beta|$

If $\alpha, \beta \in \mathcal{C}^0(S)$ define the <u>geometric intersection number</u>

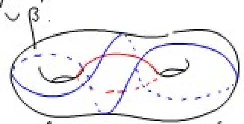
$$i(\alpha, \beta) = \min \left\{ |\alpha' \cap \beta'| \;\middle|\; \alpha' \simeq \alpha, \beta' \simeq \beta \right\}$$

<u>Bigon Criterion</u> [Epstein 1966]

(*) $|\alpha \cap \beta| = i(\alpha, \beta)$ iff $S \setminus (\alpha \cup \beta)$ has no bigons.

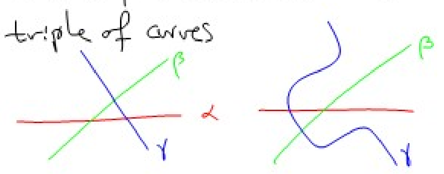
(**) Suppose $\alpha' \simeq \alpha$, $\beta' \simeq \beta$ and $|\alpha \cap \beta| = |\alpha' \cap \beta'| = i(\alpha, \beta)$

Then there is an isotopy F s.t. $F_0 = Id_S$ and $F_1(\alpha \cup \beta) = \alpha' \cup \beta'$.



<u>Exercise</u>: Read enough of Epstein 1966, or of Farb Margalit [see course website] to prove this.

NB: The second point does <u>not</u> hold for a triple of curves



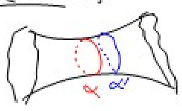
<u>Definition</u>: Say α, β <u>fill</u> S if $\forall \gamma \in \mathcal{C}^0(S)$ either $i(\gamma, \alpha) > 0$ or $i(\gamma, \beta) > 0$ (or both)

For example of this, see above.

<u>Exercise</u>: Find such filling pairs for all surfaces S.

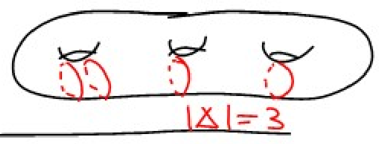
(IV) The curve complex [Harvey]

<u>Def</u>: $\Delta \subseteq \mathcal{C}^0(S)$ is a <u>multicurve</u> if $\forall \alpha, \beta \in \Delta$, $i(\alpha, \beta) = 0$.



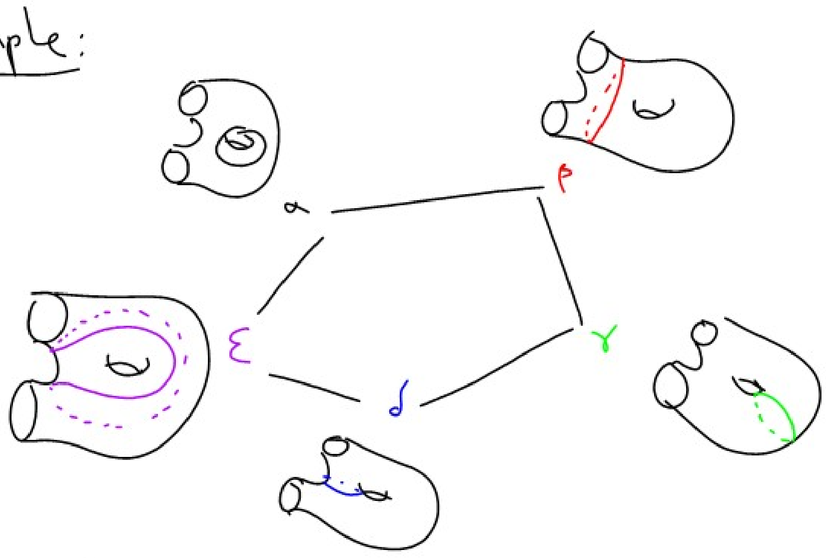
Exercise: $|\Delta| \leq \xi(S) \equiv 3g-3+n$

where $S = S_{g,n}$



$\Delta \subseteq \mathcal{C}^0(S)$ is a simplex if it is a multicurve.

Example:



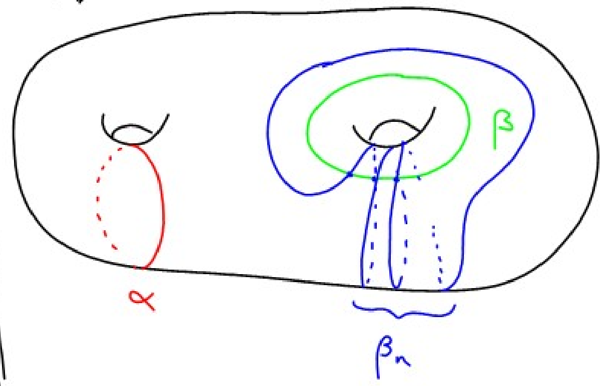
Open problem: Draw the rest of $\mathcal{C}(S_{1,2})$.

Check: $\mathcal{C}(S_{1,2})$ contains no triangles.

(V) Locally infinite

Any vertex in $\mathcal{C}(S)$ has ∞-many neighbors.

Rmk: $\mathcal{C}^0(S)$ is countable. **Prove it**



Claim: $i(\beta, \beta_n) = n$ so all β_n are distinct.

— **Exercise** —