Today's workshop is designed to help you practice computations with the derivatives involved in Green's Theorem and Stokes' Theorem.

We will use ∇ for the "del" operator, f for scalar functions, and F for vector fields. (Be sure to pronounce the last two correctly!) Here is the list of possible derivatives.

- The gradient of f, written ∇f, points in the direction of steepest ascent and measures the rate of ascent.
- The curl of F, written $\nabla \times F$, measures the rotation of the field.
- The divergence of F, written $\nabla \cdot F$, measures the expansion of the field.

In dimension two, in rectangular coordinates, we have $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$ and $F = \langle P, Q \rangle$; a vector of two scalar functions. The derivatives then become:

- $\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle f_x, f_y \rangle$.
- $\nabla \times F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \times \langle P, Q \rangle = Q_x - P_y$.
- $\nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \cdot \langle P, Q \rangle = P_x + Q_y$.

In dimension three, in rectangular coordinates, we have $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ and $F = \langle P, Q, R \rangle$; a vector of three scalar functions. The derivatives then become:

- $\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle f_x, f_y, f_z \rangle$.
- $\nabla \times F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R \rangle = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$.
- $\nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_y$.

Problem 9.1. (Warm up.) Compute the gradient of the functions $f(x, y) = x^2 + y^2$ and $g(x, y) = e^x \cos(y)$. Compute the curl and divergence of the tangential, radial, and shear fields in two dimensions. Compute the curl and divergence of the inverse tangential field $H = \frac{1}{r^2} \langle -y, x \rangle$.

Problem 9.2. Check that, in dimension two, $\nabla \times (\nabla f) = 0$ for any function f. That is, the curl of a gradient field is always zero. Begin by checking a few examples, like $f(x, y) = x^2 + y^2$ or $g(x, y) = e^x \cos(y)$. Clairaut's Theorem on page 916 may be helpful.

Problem 9.3. Check that this also holds in dimension three $\nabla \times (\nabla f) = 0$. Again, do a few examples first: like $f(x, y, z) = xyz$ or $g(x, y, z) = ye^{x^2+z}$. (The "fundamental theorem of line integrals" explains why this works – gradient vector fields are always independent of path.)

Problem 9.4. Also in dimension three, check that

$$\nabla \cdot (\nabla \times F) = 0.$$

The above three exercises can be summarized by saying "taking two derivatives yields zero."

Problem 9.5. Recall that the one variable product rule says that $(fg)_x = f_x g + f g_x$. Similar rules hold in higher dimensions. For example: suppose that f and g are both functions on $\mathbb{R}^3$. Find a formula for the gradient of the product fg solely in terms of f, g, ∇f and ∇g.

Problem 9.6. If f is a function and F is a vector field, both in the same dimension, then we can define a new vector field by scaling: $G = fF$. Derive formulas for $\nabla \times G$ (in dimensions two and three) and for $\nabla \cdot G$ in terms of f, F, and their derivatives. The formula you find should be coordinate free: f_x, P_x, and so on should not appear.

Problem 9.7. (Hard.) Here is a final product rule problem: suppose that F and G are both vector fields. Find expressions for the derivatives of $F \cdot G$ and $F \times G$ in dimensions two and three. (Compare to problem 20 on page 1136 of the book.) An important special case is the gradient of $|F|$.

Problem 9.8. (Very hard.) The definitions above for ∇f, $\nabla \times F$ and $\nabla \cdot F$ were given in rectangular coordinates. Can you find suitable definitions in polar coordinates? What about cylindrical or spherical coordinates?