Today's workshop focuses on computing integrals using the change of variables formula. Remember – the determinant of the total derivative (the Jacobian) measures the local change in length (or area, or volume) and so it shows up in the computation.

Problem 8.1. Let D be the region in $\mathbb{R}^3$ bounded by the ellipsoid $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Compute the volume of D. (If you have a hard time with this, first think about how you would compute the area bounded by an ellipse in the plane.)

Problem 8.2. Let U be the unit square in the uv–plane. That is, U has vertices at the points $(0,0),(1,0),(1,1),(0,1)$. Consider the transformation $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $S(u,v) = (u^2 - v^2, 2uv)$.

- Compute DS, the total derivative of S, and find the Jacobian $\det(DS)$.
- Carefully draw a picture of U. Label the corners and edges. Carefully draw $X = S(U)$, the *image* of U under S. Label the corresponding corners and edges. Indicate the images of typical horizontal and vertical lines.
- Compute the area of X using one-variable techniques.
- Now compute the area of X using the change of variables formula.

Problem 8.3. Let U be the rectangle in $\mathbb{R}^2$ with vertices at $(0,0),(2\pi,0),(2\pi,2\pi),(0,2\pi)$. Consider the transformation $E\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$E(u,v) = (e^{u+v}\cos(v), e^{u+v}\sin(v)).$$

- Compute DE, the total derivative of E, and find the Jacobian $\det(DE)$.
- Carefully draw and label $X = E(U)$, the image of U under E. Indicate the images of typical horizontal and vertical lines.
- Discuss how a tiny square in U is transformed by E. In particular, how does E distort the area of the tiny square? Sketch an example.
- Compute the area of X.

Problem 8.4. (See also Problem 36, page 1009 of Stewart.) The goal of this problem is to compute the improper integral $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. This is defined to be the limit $\lim_{a\to\infty} \int_{-a}^{a} e^{-x^2}\,dx$. We proceed as follows:

1. Draw a picture of the graph. (This is a version of the famous Bell Curve.)
2. Square I and change the variable of integration of the second copy of I to be y instead of x.

3. Rewrite I^2 as a multivariable integral. Give the definition of this indefinite multivariable integral. (Hint: integrate over squares with sidelength $2a$, centered at the origin. Take a limit.) Draw a picture.

4. Change to polar coordinates. *Briefly* discuss why you can take the limit of the integral over disks of radius a, instead of squares. (That is; argue why the difference of the two integrals goes to zero as $a \to \infty$.) Draw a picture.

5. Find the value of the resulting improper integral by computing the polar integral over a disk of radius a and taking a limit.

6. Take a square root to find the value of I.

Problem 8.5. (Found on an exam review of Prof. Greenfield's.) The constraint $x^4 + x^2y^2 + 2y^4 + z^4 = 1$ defines a closed and bounded set in $\mathbb{R}^3$ and thus the function $f(x, y, z) = xyz$ attains a maximum value on that set. What is this maximum value? Be sure to analyze carefully and completely any system of equations you solve.