The point of today's workshop is to become familiar with computing with matrices and the total derivative. We'll discuss the geometric meaning of these objects next class.

Problem 7.1 (Matrices). An $n \times m$ *matrix* is a rectangular array, with n rows and m columns. For example:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}.$$

A matrix need not be square. For example:

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}, \ F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Suppose that X is a $p \times n$ matrix and Y is a $n \times m$ matrix. We define a new $p \times m$ matrix $X \cdot Y$ where the ij^{th} entry is the dot product of the i^{th} row of X with the j^{th} column of Y. For example:

$$B \cdot C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}.$$

- Is $A \cdot C = C \cdot A$?
- Compute $C^2 = C \cdot C$. Compute $C^3 = C^2 \cdot C$. Compute all powers of C: C^2, C^3, C^4, and so on.
- Can you compute all powers of A? Of B?
- Determine which of A, B, C, D, E, F (as above) can be multiplied. For those which can be multiplied, decide the number of rows and columns of the product.

Problem 7.2. There are several important families of two-by-two matrices. We have the *rotation* matrices,

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

the *shear* matrices

$$S_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

and the *hyperbolic* matrices

$$H_k = \begin{bmatrix} k & 0 \\ 0 & k^{-1} \end{bmatrix}.$$

These names come from thinking of R_θ, S_t, H_k as transformations of $\mathbb{R}^2$.

- Pick a nice value of θ (say, $\pi/3$) and compute the product of R_θ with various 2×1 matrices (ie vectors in $\mathbb{R}^2$).
- Do the same for S_t and H_k.
- Explain with words and pictures how R_θ, S_t, and H_k transform the plane.
- Now compute $R_\theta \cdot R_\tau$, $S_s \cdot S_t$, and $H_k \cdot H_\ell$. Check that the answers you get match the geometric explanation you gave above.

Problem 7.3. The most important quantity associated to a two-by-two matrix is its *determinant*:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Notice that this is identical to the cross product of the rows of the matrix. The determinant records how the matrix expands or contracts area.

- Compute the determinant of all two-by-two matrices given above (including their powers).
- Find a relationship between $\det(X)$, $\det(Y)$, and $\det(X \cdot Y)$. Check algebraically that the relationship holds for any X and Y. Can you explain this relation geometrically?

Problem 7.4 (Total derivatives). The total derivative of a transformation is also a matrix. Find the total derivative, and the determinant of the total derivative, for each of the following:

- $M(x, y) = (x - y, x + y)$.
- $Q(u, v) = (u^2 - v^2, 2uv)$.
- $P(r, \theta) = (r\cos(\theta), r\sin(\theta))$.
- $E(x, y) = (e^{x+y}\cos(y), e^{x+y}\sin(y))$.