Please work in groups of two or three. Please explain all answers carefully.

Problem 5.1 (Warm-up). For each of the following functions, plot enough of the gradient vectors to get a feel for the *gradient vector field*: the function

$$(x, y) \mapsto \left\langle \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right\rangle$$

which takes in a point in the plane and spits out a vector based at that point. On the same xy–plane, sketch the level sets. Verify visually that the gradient is perpendicular to the level sets.

- $f(x, y) = x$.
- $f(x, y) = x^2$.
- $f(x, y) = x^2 + y^2$.
- $f(x, y) = -x^2 - y^2$.
- $f(x, y) = x^2 - y^2$.
- $f(x, y) = xy$.

Problem 5.2 (Warm-up). Sketch the following vector fields.

- $\mathbf{v}(x, y) = \langle 1, 0 \rangle$, a *constant* field.
- $\mathbf{v}(x, y) = \langle 0, 1 \rangle$.
- $\mathbf{v}(x, y) = \langle x, y \rangle$, an outward *radial* field.
- $\mathbf{v}(x, y) = \langle -x, -y \rangle$, an inward radial field.
- $\mathbf{v}(x, y) = \langle -y, x \rangle$, a *tangential* field.
- $\mathbf{v}(x, y) = \langle y, x \rangle$, a *hyperbolic* field.
- $\mathbf{v}(x, y) = \langle x, 0 \rangle$.
- $\mathbf{v}(x, y) = \langle y, 0 \rangle$, a *shear* field.

Which of these vector fields is/is not the gradient of some function? Explain your reasoning.

Problem 5.3. Consider the function $f(x, y) = x^3 - \frac{3}{2}x^2 + \frac{3}{2}y^2$.

- Give a sketch of the graph of f. It may be useful to begin by drawing the xz–traces. (That is, cross sections parallel to the xz–coordinate plane.)
- Find all critical points of f: points (x, y) where the gradient ∇f vanishes. Does this agree with the "special points" of your graph?
- Sketch the gradient vector field ∇f. Be careful near the critical points.
- The *Hessian* of f is the matrix of second partial derivatives:

$$H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

  This measures the rate of change of the gradient. Now compute

$$\det(H_f) = f_{xx}f_{yy} - f_{xy}f_{yx},$$

  the *determinant* of H_f, at the critical points of f. Check that your results agree with the "Second Derivatives Test" on pages 954-5 of Stewart. Explain what is going on using your sketch of the gradient.

Problem 5.4. Consider the function $f(x, y) = x^3 + y^3 - \frac{3}{2}(x^2 + y^2)$.

- Compute the gradient of f. Find all critical points of f by finding all solutions of $\nabla f = (0, 0)$.
- Now sketch the gradient along the lines $y = 0, y = 1, x = 0, x = 1$. Fill in as much of the gradient vector field as you can around the critical points. How is the gradient behaving in a small region around each of the critical points?
- What does the graph of the function look like at those places? Check your work by computing the Hessian and using the second derivative test.
- Now find level curves for the graph of f. Give a sketch of the graph of the function. Again, it may help to look at the xz–traces.