Please work in groups of two or three. Please explain all answers carefully.

Suppose $p(t)$ is a space curve. The following formulas may be useful (all derivatives are taken with respect to t):

$$\mathbf{T} = \frac{p'}{|p'|}, \quad \mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}, \quad \mathbf{B} = \mathbf{T} \times \mathbf{N},$$

$$\kappa = \frac{|\mathbf{T}'|}{|p'|}, \quad \frac{\mathbf{B}'}{|p'|} = -\tau \mathbf{N}, \quad s(t_0) = \int_a^{t_0} |p'(t)|\, dt.$$

Problem 4.1 (Warm-up). Let $p(t) = (\cos(t), \sin(t), t)$. Find a parametrization of the tangent line to p at time $t = \pi/2$. Give a sketch.

Problem 4.2. Suppose that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are vector valued functions. Use coordinates to verify the product rules:

- $(\mathbf{u} \cdot \mathbf{v})' = (\mathbf{u}' \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{v}')$ and
- $(\mathbf{u} \times \mathbf{v})' = (\mathbf{u}' \times \mathbf{v}) + (\mathbf{u} \times \mathbf{v}')$.
- Check that if $|\mathbf{u}(t)| = 1$ for all t then $\mathbf{u} \cdot \mathbf{u}' = 0$. Can you give an example of such a function?

Problem 4.3 (Warm-up). Compute the *Frenet apparatus* of the following space curves. (Ie, compute $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, κ, and τ.) Also compute the arclength, if you can. It will help to think geometrically.

- $p(t) = (\cos(t), \sin(t), t)$. (It may help to draw helix on the cylinder $C : x^2 + y^2 = 1$, and then "unroll" the cylinder.)
- $q(t) = (\cos(2t), \sin(2t), 2t)$.
- $r(t) = (\cos(t), \sin(t), 2t)$.
- $c(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}, 0\right)$.

Problem 4.4 (Stewart, p. 870). We will now verify some of the claims made in class.

- Show that $\mathbf{B}'$ is perpendicular to $\mathbf{B}$. (Likewise, $\mathbf{T}'$ is perpendicular to $\mathbf{T}$ and $\mathbf{N}'$ is perpendicular to $\mathbf{N}$.)
- Show that $\mathbf{B}'$ is perpendicular to $\mathbf{T}$.
- Show that $\mathbf{N}'/|p'| = -\kappa \mathbf{T} + \tau \mathbf{B}$.

Problem 4.5 (Repeat). Suppose we are given vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$. Define

$$P(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \{r\mathbf{u} + s\mathbf{v} + t\mathbf{w} \mid 0 \leq r \leq 1,\ 0 \leq s \leq 1,\ 0 \leq t \leq 1\}.$$

This is a *parallelepiped*. Draw a picture of $P(\mathbf{i}, \mathbf{j}, \mathbf{k})$. Draw a picture of $P(\mathbf{i}, \mathbf{i}+\mathbf{j}, \mathbf{i}+\mathbf{k})$. Compute the volumes.

Now, what does a "general" parallelepiped look like? How would you find its volume?

Problem 4.6 (Hard). Let $p(t) = (t, t^2, t^3)$ be the *twisted cubic*. Suppose that $a, b, c \in \mathbb{R}$ are distinct real numbers. Show that the points $A = p(a)$, $B = p(b)$, and $C = p(c)$ are not co-linear. (That is, do not lie on a line.)