Math 311: Section 3.
Solutions for some questions on the first midterm.

Problem 1.1 (20 points)**.** Suppose that a, b, c, and d are positive real numbers where $a < b$ and $c < d$.

(1) Prove directly from the ordered field axioms that $ac < bc$. You may assume without proof that $(\forall x \in \mathbb{R}, x \cdot 0 = 0)$ and $(\forall x \in \mathbb{R}, x \cdot (-1) = -x)$.
(2) Using the above prove that $ac < bd$.

Solution 1.1. Suppose that a, b, c, d are as given in the hypothesis. We can make the following deductions:

Statement	Justification
$a < b$	Hypothesis
$a + (-a) < b + (-a)$	Order distributivity
$0 < b + (-a)$	Additive inverse
$0 < c$	Hypothesis
$0 < (b + (-a))c$	Positivity
$0 < bc + (-a)c$	Distributivity
$0 < bc + (-1)ac$	Hypothesis, Assoc.
$0 < bc + (-(ac))$	Hypothesis
$ac < bc + ac + (-ac)$	Order dist., Assoc., Commut.
$ac < bc$	Additive inverse

We are now equipped to prove that $ac < bd$:

Statement	Justification
$ac < bc$	By the above claim.
$cb < db$	By the above claim.
$bc < bd$	Commutivity twice.
$ac < bd$	Transitivity

Problem 1.5 (20 points)**.** Decide whether or not the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converges. Prove your answer is correct.

Solution 1.5. We begin by claiming, for all $n \in \mathbb{N}$, that $\frac{n}{n^2+1} \geq \frac{1}{2n}$. To see this, note that $2n^2 = n^2 + n^2 \geq n^2 + 1$. Cross-dividing gives the result.

Now note that if $\sum \frac{1}{2n}$ diverges then so does $\sum \frac{n}{n^2+1}$ by the comparison test. So we have reduced the problem to showing that $\sum \frac{1}{2n}$ diverges.

For a contradiction, suppose that $\sum \frac{1}{2n}$ converges. Then so does $\sum \frac{2}{2n}$, by the Algebraic Limit Theorem for Series (2.7.1). However, this last is the harmonic series and we have proved in class that it does *not* converge. The contradiction is obtained and the proof is complete. □

Problem 1.6. (Extra Credit – attempt only after double checking the rest of the exam.) Decide whether or not the series $\sum_{n=1}^{\infty} \frac{\log_2(n)}{n^2}$ converges. Justify your answer.

Solution 1.6. Recall the definition of $\log_2(n)$: it is the number x so that $2^x = n$. The only property of $\log_2$ we will need is that $\log_2(2^k) = k$, for any natural number k. Recall also, for all $k \in \mathbb{N}$, that $k^2 \leq 2^k$. (This was proven by most of you for a workshop.) Taking the square-root of both sides we deduce the estimate $k \leq 2^{k/2} = (2^{1/2})^k$. (To be strictly correct here: we are using the fact that $0 < x < y$ implies that $0 < \sqrt{x} < \sqrt{y}$. That is, the square-root function is an *increasing* function.)

By the Cauchy Condensation Test the series $\sum \frac{\log_2(n)}{n^2}$ converges if and only if the series $\sum \frac{2^k \log_2(2^k)}{(2^k)^2}$ converges. A bit of algebra shows that this last series is identical to the series $\sum \frac{k}{2^k} = \sum \frac{k}{(2^{1/2})^k (2^{1/2})^k}$.

Note that $\frac{k}{(2^{1/2})^k (2^{1/2})^k} \leq \frac{1}{(2^{1/2})^k}$ using the estimate of the first paragraph. Finally, the series $\sum \frac{1}{(2^{1/2})^k}$ converges, since it is geometric with ratio $\frac{1}{2^{1/2}} < 1$. We deduce from the comparision test that the series $\sum \frac{k}{2^k}$ also converges, and we are done. $\square$