Math 311: Section 3.

Workshop 8: One mean theorem.

Discuss several and then do any one of these problems.

Problem 8.1. Prove that the function $f(x) = x^3 + 3x - 2$ has exactly one real root.

Problem 8.2. Prove that the function $f(x) = x^3 - 3x - 3$ has exactly one real root. (Your techniques may need to change a bit.)

Problem 8.3. Carefully prove the inequality

$$4x^3 - 3x^4 \leq 1.$$

Problem 8.4. Show for all positive real numbers x that

$$\frac{x}{1+x} \leq \ln(1+x) \leq x.$$

(Hint: draw a few pictures.)

Problem 8.5. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable everywhere and that $f'(x)$ is an increasing function. Prove that for all $x, c \in \mathbb{R}$ the following inequality holds:

$$f(x) \geq f(c) + f'(c)(x - c).$$

Any function with this property is called a *convex* function. (Again, drawing a few examples will help.)

Problem 8.6. And now for something completely different – here is a nice problem taken from a friend's calculus book. Define a function $f\colon [0,1] \to [0,1]$ which takes the real number $r = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$ to $f(r) = \sum_{n=1}^{\infty} \frac{d_n}{10^{2n}}$. The numbers d_n lie in the set $\{0, 1, 2, 3, \ldots, 9\}$. The series given for r sums to $0.d_1d_2d_3d_4\ldots$ This is called the *decimal expansion* of r. (Here we do *not* allow all but finitely many of the d_n's to be the numeral 9.)

Here are two questions about f: What is the limit as you approach 1 from the left? What is the limit as you approach 1/2 from the left? From the right? What are the points of discontinuity of f?