Math 311: Section 3.

Workshop 6: Cantor set, Cantor set.

Problem 6.1. This problem sketches a proof of the fact that $C + C = \{x + y \mid x, y \in C\} = [0, 2]$, where C is the Cantor set. Begin by showing that $C + C \subset [0, 2]$. The other inclusion is more difficult and is broken into two steps:

- Prove that, for all $s \in [0, 2]$, there are a pair of elements x_n and y_n in the set C_n satisfying $x_n + y_n = s$.
- Keeping in mind the sequences (x_n) and (y_n) do not necessarily converge, show how they can nevertheless be used to produce points $x, y \in C$ so that $x + y = s$.

(As a hint for the last step: if $x_n \in C_n$ then $x_n \in [0, 1]$ for all n. Thus the sequence (x_n) is bounded.)

Problem 6.2. Fix a set of distinct real numbers, indexed by the natural numbers, $A = \{a_n\}_{n=1}^{\infty}$. Define

$$f_A(x) = \begin{cases} 0 & x \notin A \\ 1/n & x = a_n. \end{cases}$$

and show that f_A is discontinuous at exactly the points of A. (Hint: recall our discussion of Thomae's function in class. You may use without proof the fact that the complement of A, A^c, is dense in $\mathbb{R}$.)

Recall that a function $g \colon \mathbb{R} \to \mathbb{R}$ is *increasing* if $a \leq b$ implies that $g(a) \leq g(b)$.

Problem 6.3. Consider a set of points $A = \{a_n\}$ as above. This problem finds an increasing function $g_A \colon \mathbb{R} \to \mathbb{R}$ which is not continuous at exactly the points of A. Here is the construction:

- Define the *step function* at a to be

$$S_a(x) = \begin{cases} 0 & x < a \\ 1 & x \geq a. \end{cases}$$

  Check that S_a has exactly one point of discontinuity, at a. Give a sketch.
- Consider the function

$$g_m = \sum_{n=1}^{m} \frac{S_{a_n}}{2^n}.$$

  How many discontinuities does g_m have? Supposing that $a_n = n$, what does g_4 look like? Supposing that $a_n = 1/n$, what does g_4 look like? Give sketches.
- Consider the function $g_A \colon \mathbb{R} \to \mathbb{R}$ defined at x by:

$$g_A(x) = \lim_{m \to \infty} g_m(x).$$

  That is, for any real number x we define $g_A(x)$ to be the limit of the series $(g_m(x))_{m=1}^{\infty}$. Show that g_A is defined for all real numbers x. Show that g_A is continuous at $a \in \mathbb{R}$ if and only if a is not an element of the set $\{a_n\}$.

The above construction contrasts nicely with the following theorem (Exercise 4.6.5 in the book): Any increasing function has at most countably many discontinuities.

Problem 6.4. Find a function h_C, defined on all of $\mathbb{R}$ which is not continuous exactly at the points of the Cantor set. (Hint: The Cantor set C is uncountable. Thus, according to Exercise 4.6.5 of the book, it cannot occur as the discontinuity set of a monotone function. From this we deduce that h_C must *oscillate*; that is, the output of h_C must vary up and down a great deal. Here is another hint: there is such a function h_C which outputs only the values zero and one.)

Question 6.5. Can you find a function having exactly the set of irrational numbers as its discontinuity set?