Math 311: Section 3.

Workshop 1.5: π is irrational.

A student suggested in class that π might be irrational. Here is a sketch of a proof, stolen from Ivan Niven's article in the Bulletin of the American Mathematical Society [1947]. Provide the details.

Theorem 0.1. *The number π is irrational.*

Proof. Suppose, for a contradiction, that $\pi = a/b$ where a and b are positive integers. Let

$$f(x) = \frac{x^n(a-bx)^n}{n!} = \frac{b^n x^n(\pi - x)^n}{n!}$$

where n is some large positive integer to be chosen later.

Problem 0.2. Show that $f^{(j)}(0)$ and $f^{(j)}(\pi)$ are integers, for all non-negative integers j. Here $f^{(j)}$ denotes the j^{th} derivative of f. (A helpful thing to check: $f(x) = f(\pi - x)$.)

Problem 0.3. Using the above show that, for any choice of $n \in \mathbb{N}$, the integral $\int_0^\pi f(x)\sin(x)\,dx$ is an integer. (Hint: repeated integration by parts.)

Problem 0.4. On the other hand, for n sufficently large, show that the integral above lies strictly between zero and one. (Hint: What is the maximum value of $f(x)$ in the interval $[0, \pi]$?)

This is a contradiction. We are done. $\square$

Problem 0.5. Why does this proof work? What is the "idea" behind it?

Problem 0.6. Prove that e is irrational. (Note that e is very different from π, so perhaps a different idea will be necessary.)