Saul Schleimer
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Puzzling the 120–cell

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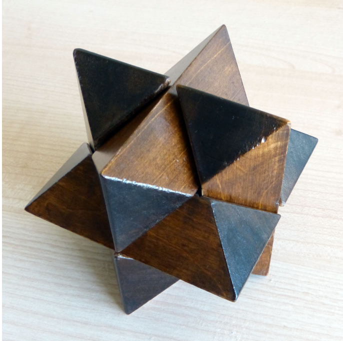
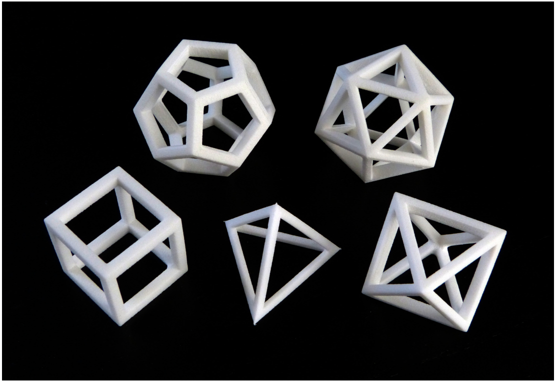
Burr puzzles

The goal of a burr puzzle is to assemble a number of “notched sticks” into a single object.



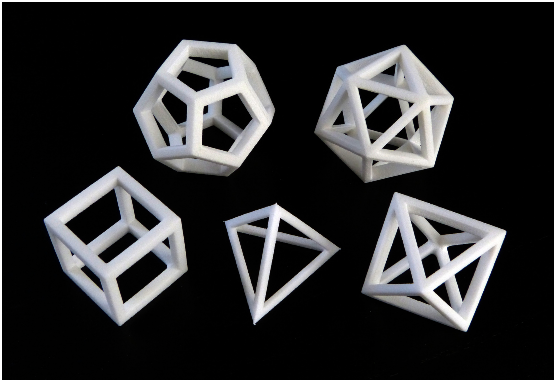
In this talk, I will describe Quintessence, a family of burr puzzles based on the 120–cell.

Platonic solids



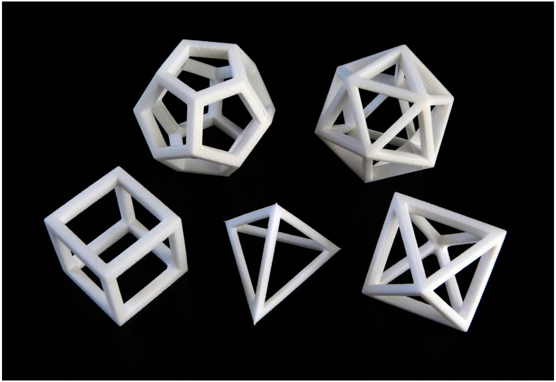
The platonic solids probably predate civilization, probably predate mathematics, and certainly predate Plato.

Platonic solids



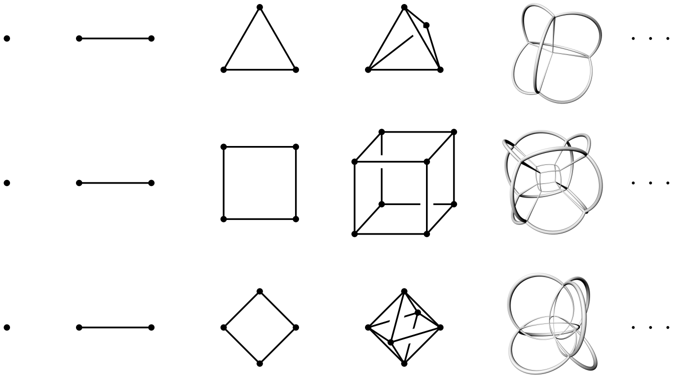
These are all of the regular polytopes in dimension three. In general, the boundary of a regular polytope is tiled by identical regular polytopes of one dimension lower. So the platonic solids are tiled by regular polygons.

Platonic solids



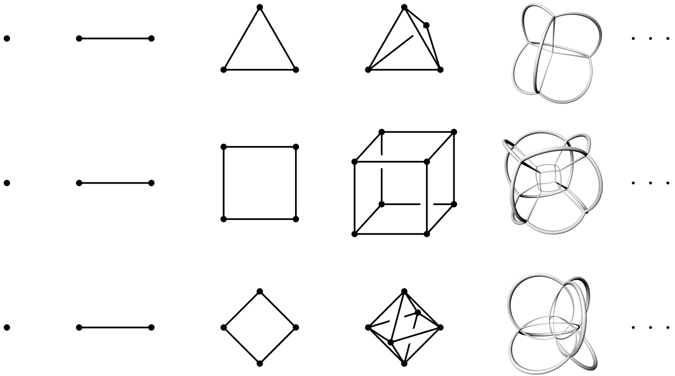
There are four infinite families of regular polytopes. The first family lives in dimension two: the triangle, the square, the pentagon, the hexagon, the septagon, and so on.

Regular polytopes



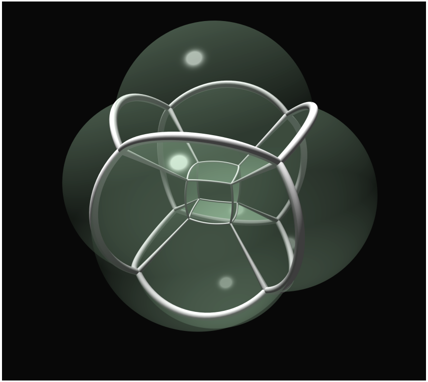
The other three families stretch across the dimensions. Here we see the simplices, the cubes, and the cross-polytopes. Again, notice that the bounary of the n–dimensional cube is tiled by copies of the $(n-1)$–dimensional cube.

Regular polytopes



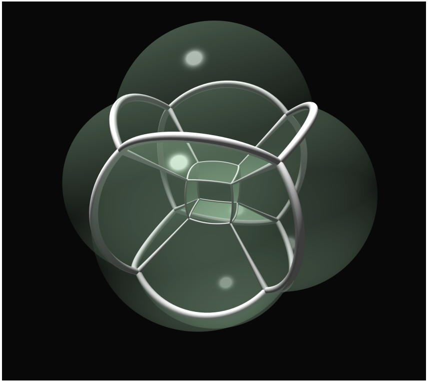
After the polygons, simplices, cubes, and cross-polytopes, there are only five regular polytopes left. The isocahedron and dodecahedron in dimension three, and the 24–cell, 120–cell, and 600–cell in dimension four.

Hypercube



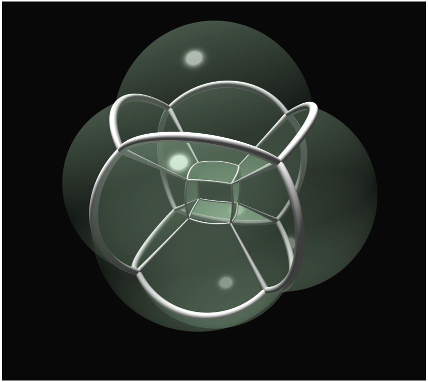
Here is the 4–cube again, variously called the 8–cell, hypercube, or tesseract. It has 16 vertices, 32 edges, 24 squares, and, as advertised, 8 cubes.

Hypercube



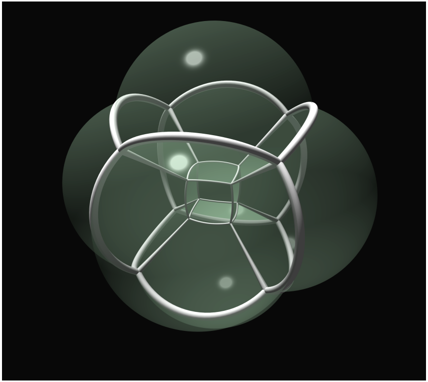
Actually, this is not the hypercube. This is the *boundary* of the hypercube.

Hypercube



Actually, it isn't *all* of the boundary of the hypercube — we had to remove a point.

Hypercube

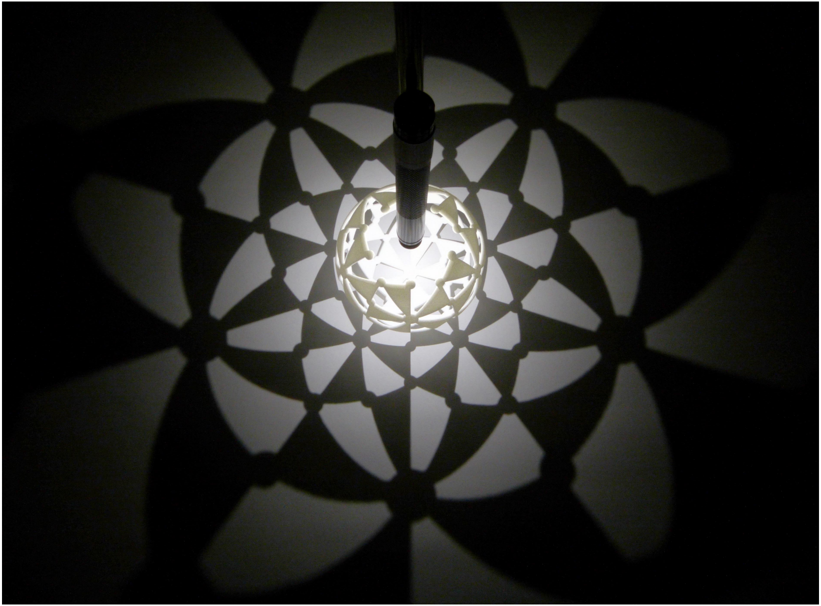


To explain why this hypercube is “curvy”, we first drop down a dimension.

Stereographic projection

DEMONSTRATION

Stereographic projection



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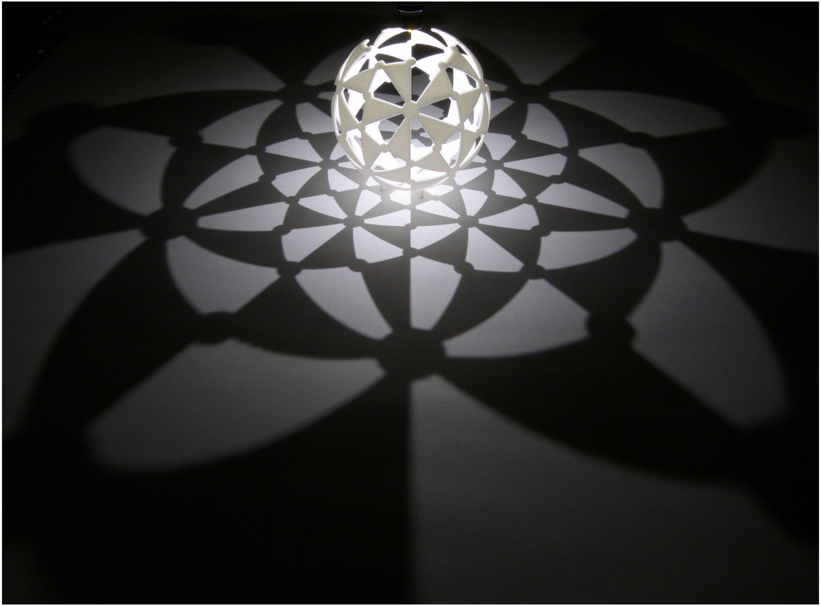

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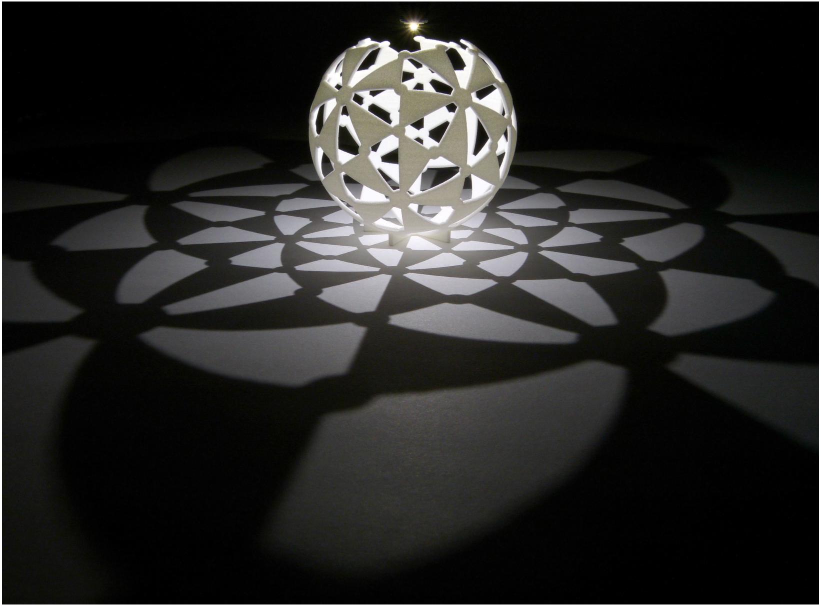
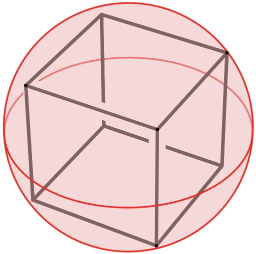


Projecting a cube from $\mathbb{R}^3$ to S^2 to $\mathbb{R}^2$



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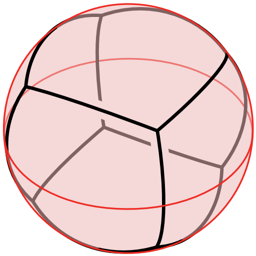

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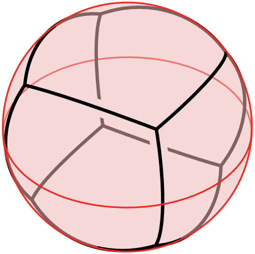

Radial projection

$$\mathbb{R}^3 \setminus \{0\} \to S^2$$

$$(x, y, z) \mapsto \frac{(x, y, z)}{|(x, y, z)|}$$

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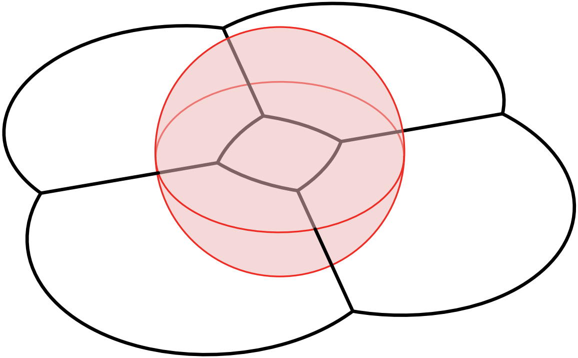

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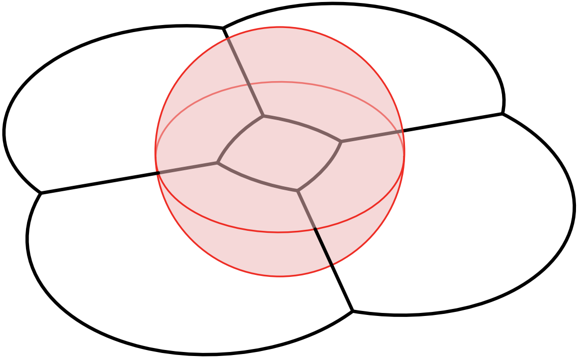

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Stereographic projection

$$S^2 \setminus \{N\} \to \mathbb{R}^2$$

$$(x, y, z) \mapsto \left(\frac{x}{1 - z}, \frac{y}{1 - z}\right)$$

Stereographic projection

In general, stereographic projection maps from $S^n \smallsetminus \{N\}$ to $\mathbb{R}^n$.

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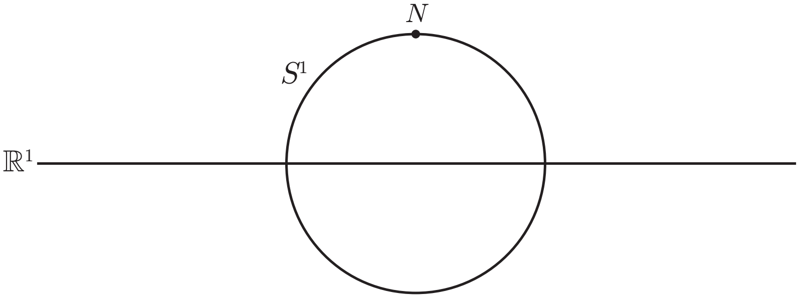

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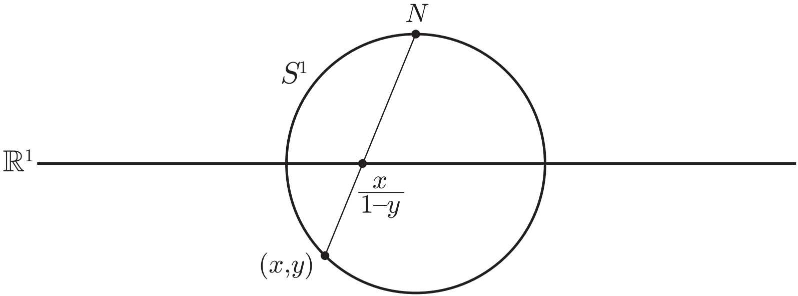

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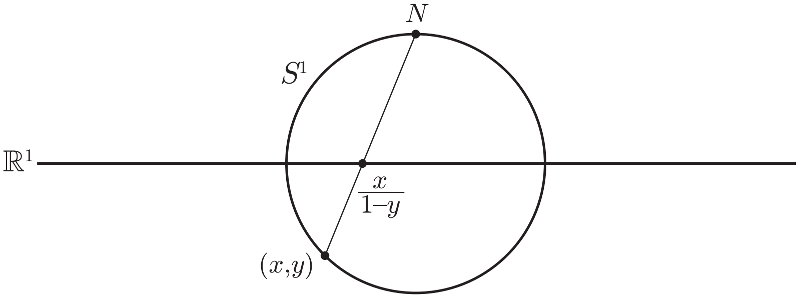
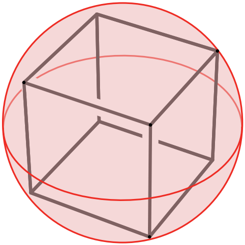


This is *also* a cross-section of stereographic projection for $n > 1$.

Thickening the edges



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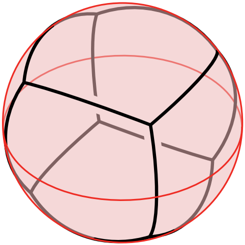

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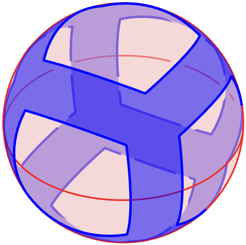

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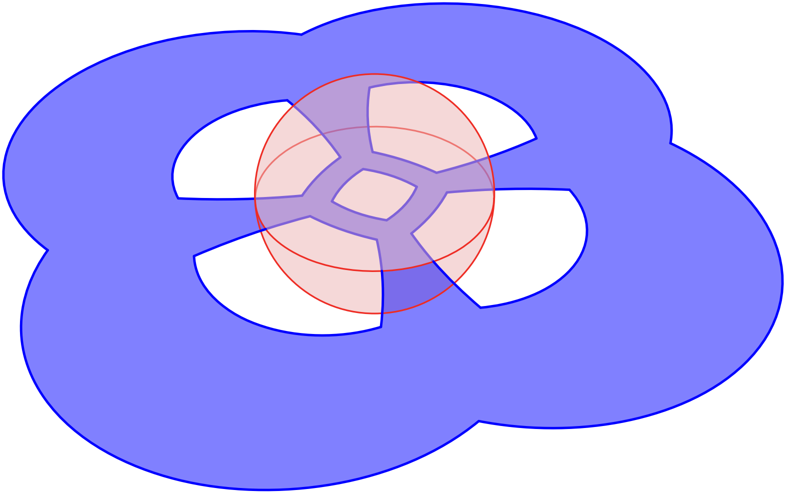

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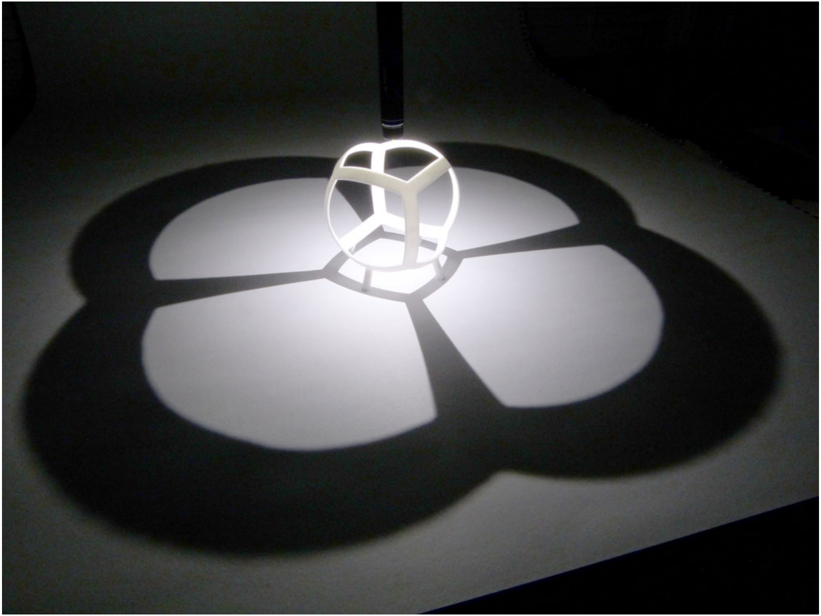

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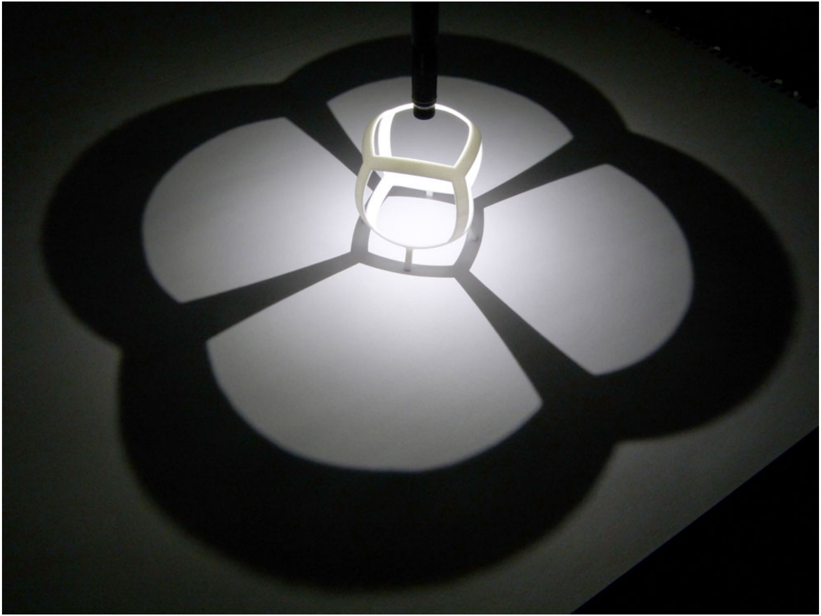
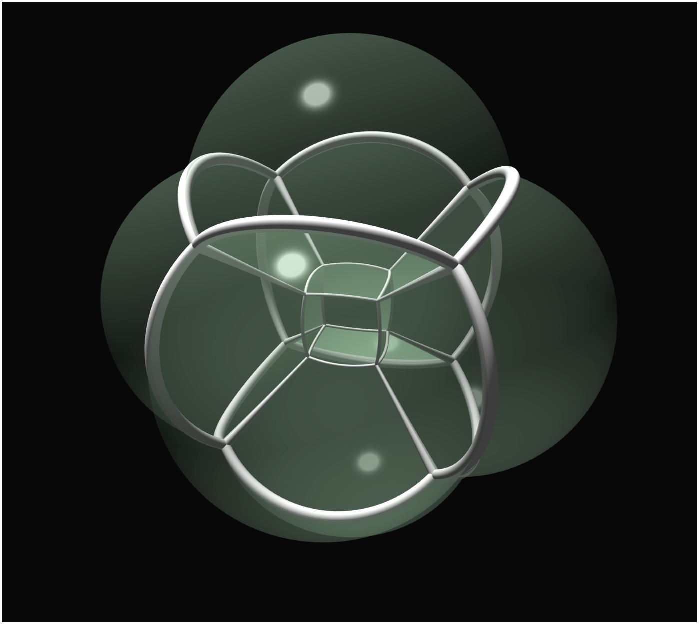
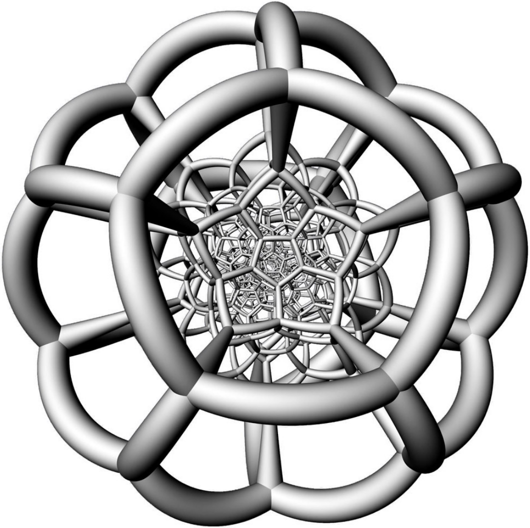


Hypercube, redux



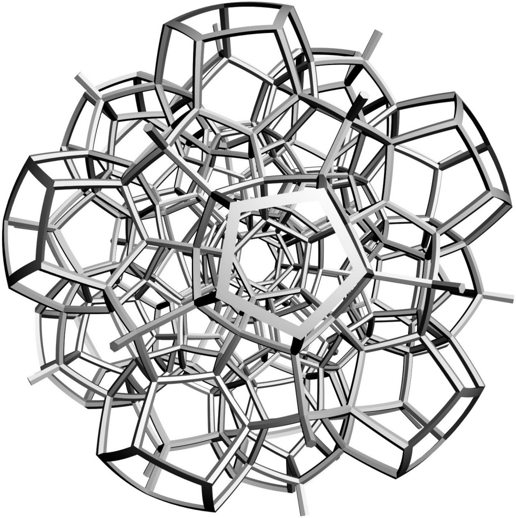
120–cell

Here is the (cell-centered) projection of the 120-cell; it has dodecahedral symmetry in $\mathbb{R}^3$.



Half 120–cell

Here is a cut-away version – we cut along the unit sphere to show the inner half of the 120–cell. Again it has dodecahedral symmetry in $\mathbb{R}^3$.



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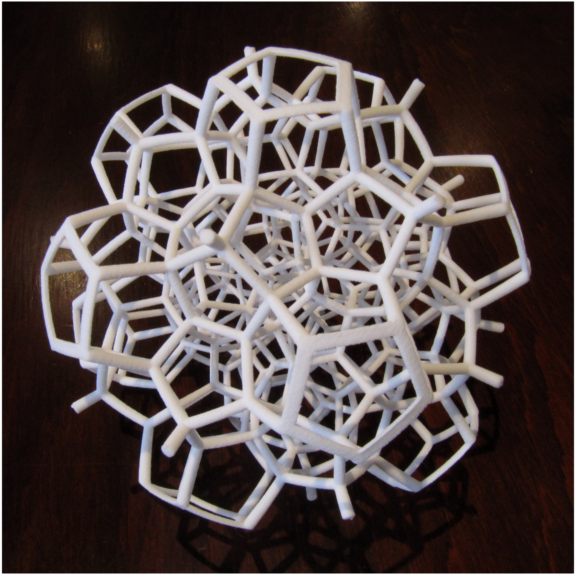

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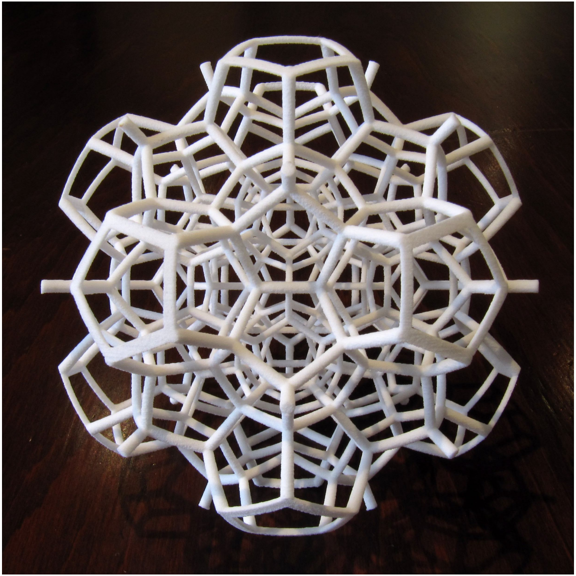

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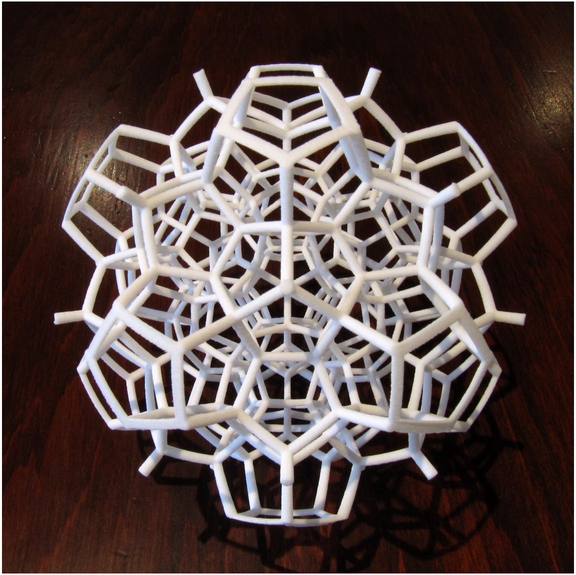

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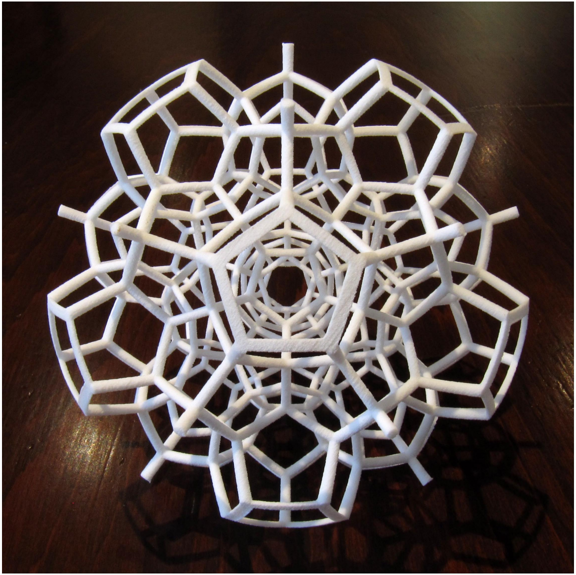

Spherical layers in the 120–cell

A first way to understand the 120–cell is to look at layers of dodecahedra at a fixed distance from the central dodecahedron.

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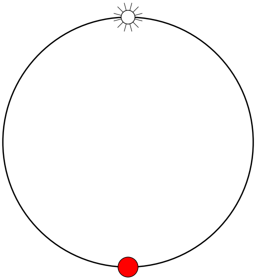
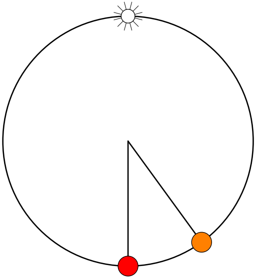


Spherical layers in the 120–cell

A first way to understand the 120–cell is to look at layers of dodecahedra at a fixed distance from the central dodecahedron.

- 1 central dodecahedron
- 12 dodecahedra at distance $\pi/5$



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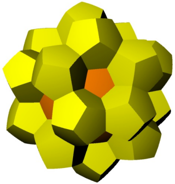

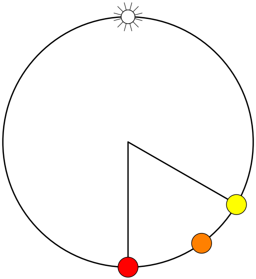

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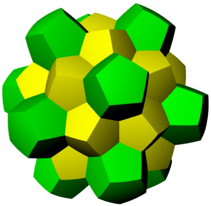
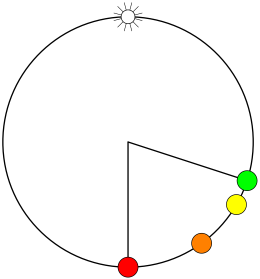
- 12 dodecahedra at distance $2\pi/5$





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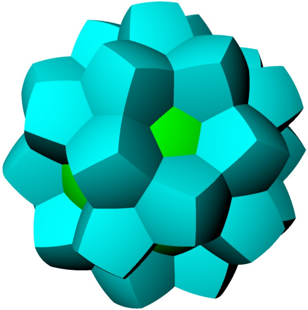

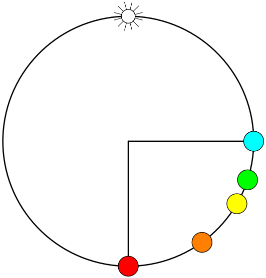

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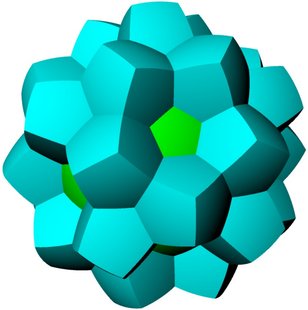
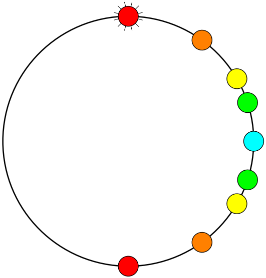
The pattern is mirrored in the last four layers.

$1+12+20+12+30+12+20+12+1 = 120$





Hopf fibers in the 120–cell

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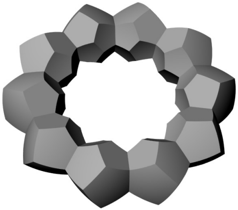
Each fiber is a "ring" of 10 dodecahedra.



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Each fiber is a "ring" of 10 dodecahedra.

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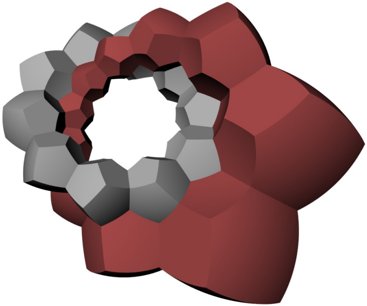

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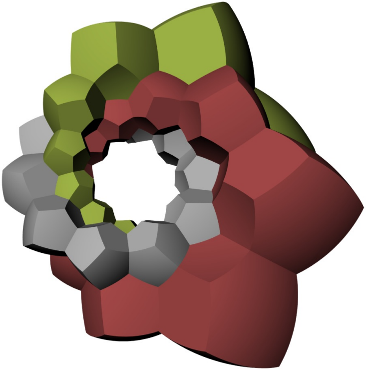

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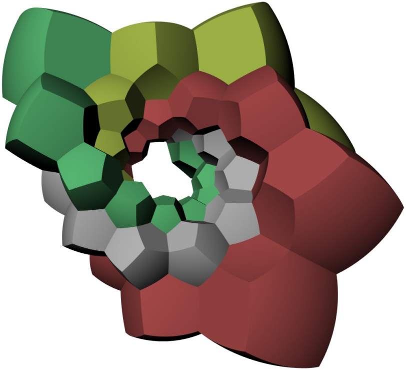

Hopf fibers in the 120–cell

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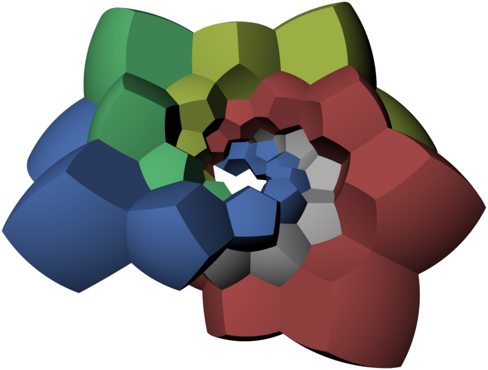
Each fiber is a "ring" of 10 dodecahedra.

The rings wrap around each other.

Each ring is surrounded by five others.



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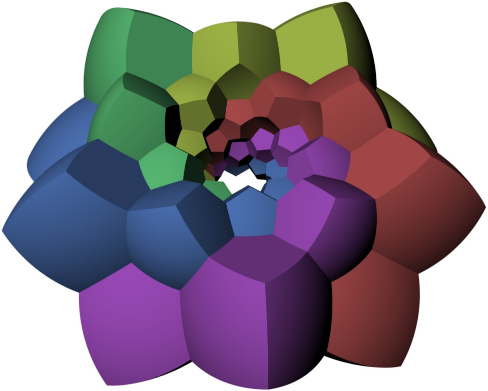

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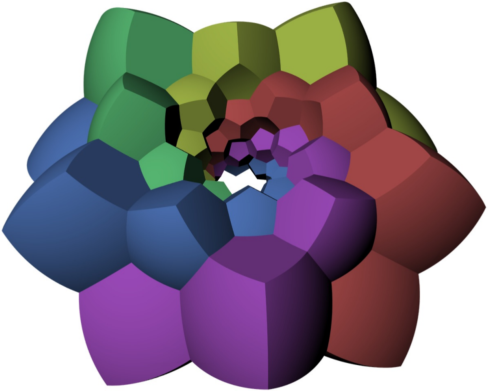

These six rings make up half of the 120–cell. The other half consists of five more rings that wrap around these, and one more ring "dual" to the original grey one.

$$1 + 5 + 5 + 1 = 12 = 120/10$$

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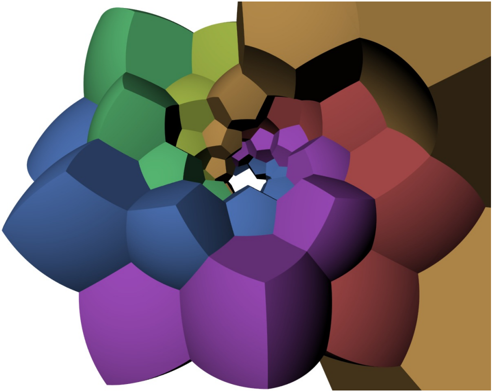

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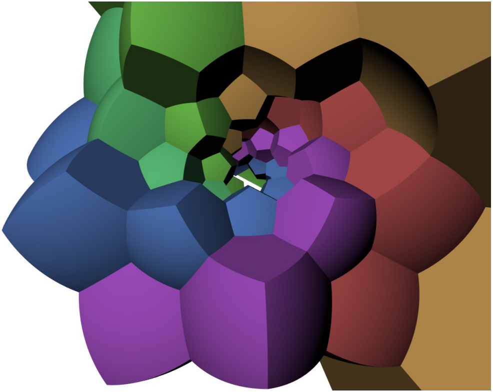
The rings wrap around each other.

Each ring is surrounded by five others.



These six rings make up half of the 120–cell. The other half consists of five more rings that wrap around these, and one more ring "dual" to the original grey one.

$$1 + 5 + 5 + 1 = 12 = 120/10$$

Hopf fibers in the 120–cell

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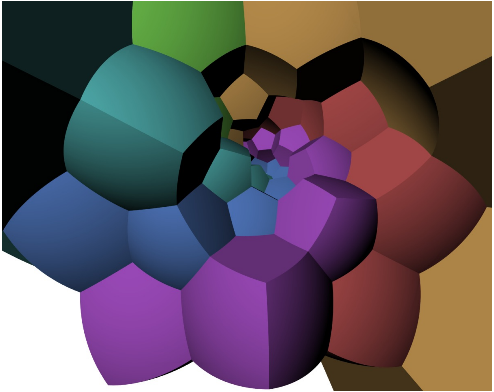

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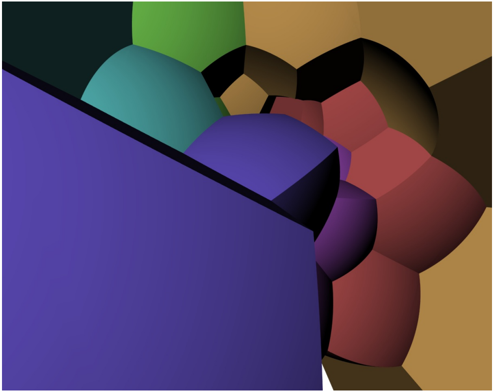

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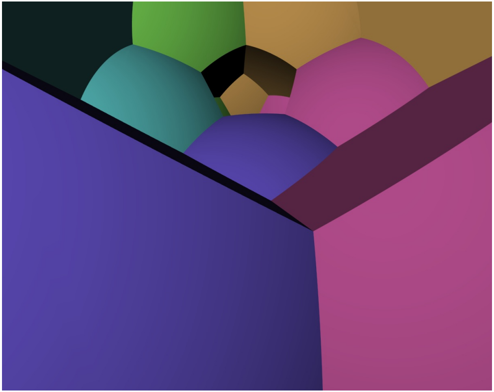

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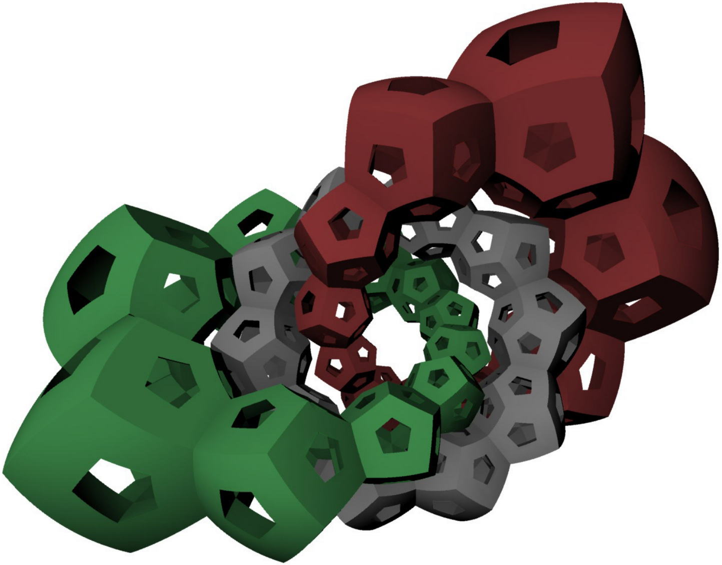
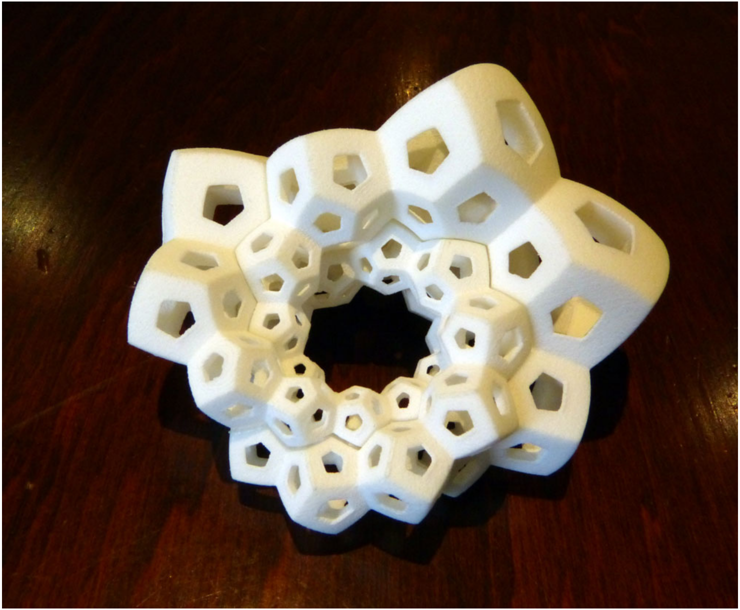
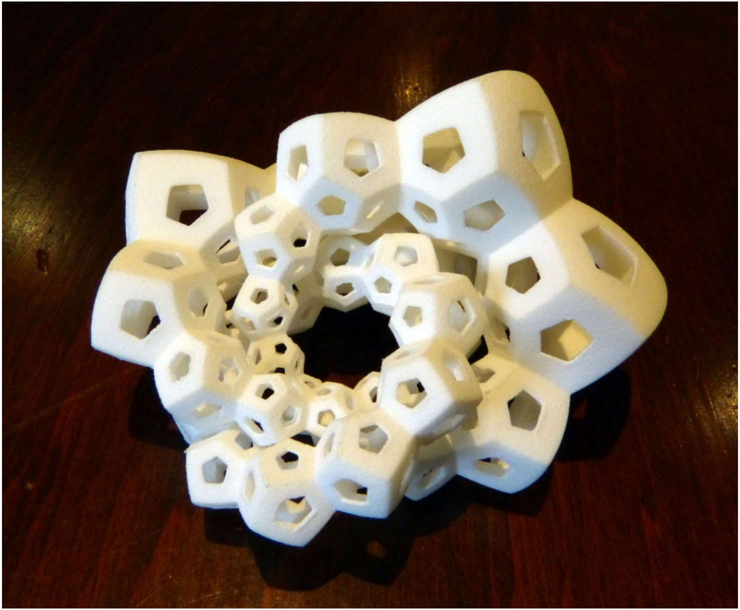
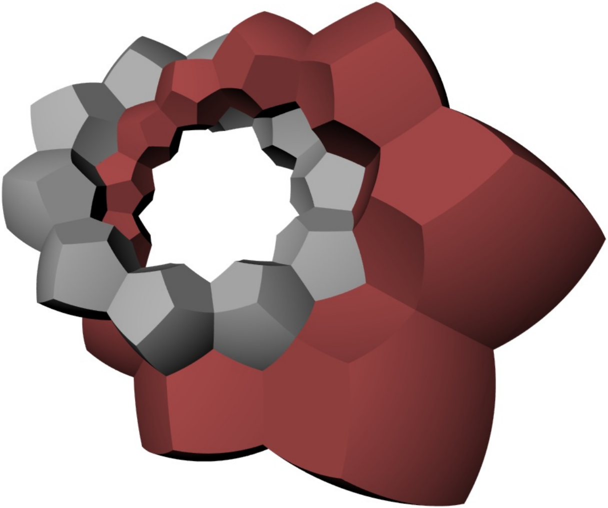
We wanted to 3D print all six of the inner rings together; it seems this cannot be done without them touching each other. (Parts intended to move must not touch during the printing process.)



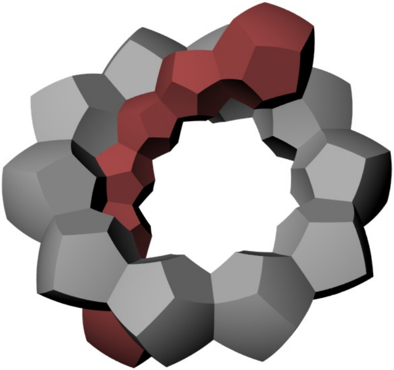




To print all five we use a trick...



To print all five we use a trick... don't print the whole ring. We call part of a ring a rib.



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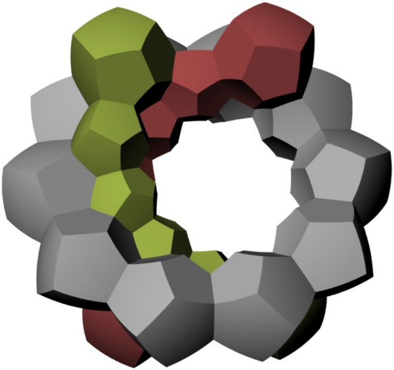

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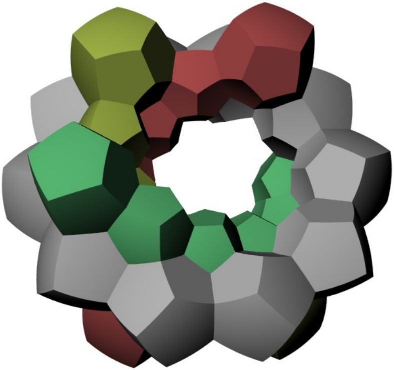

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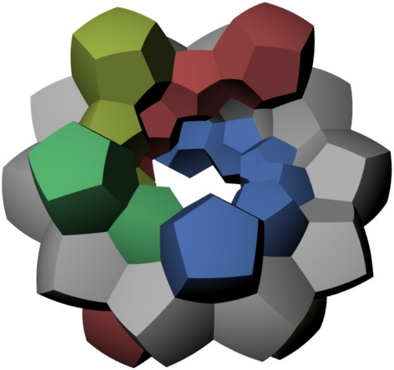

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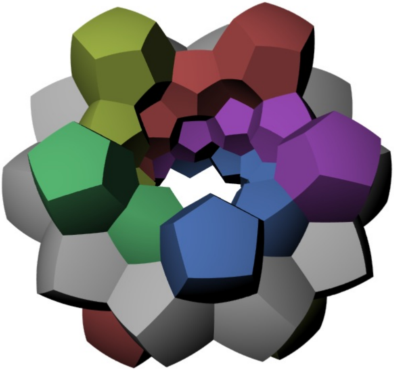

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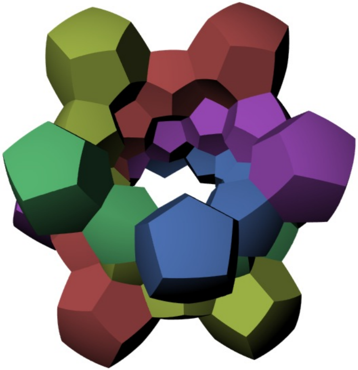

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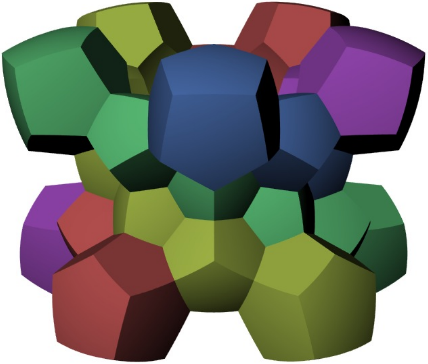
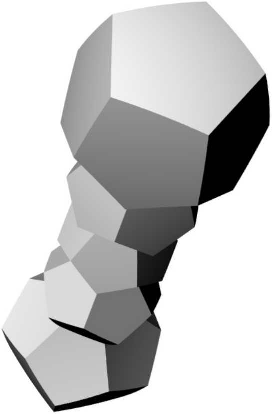


Dc30 Ring puzzle

Another decomposition, with even shorter ribs.



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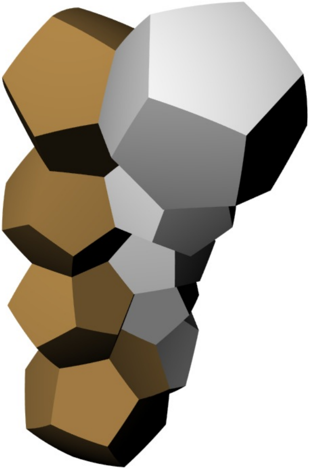

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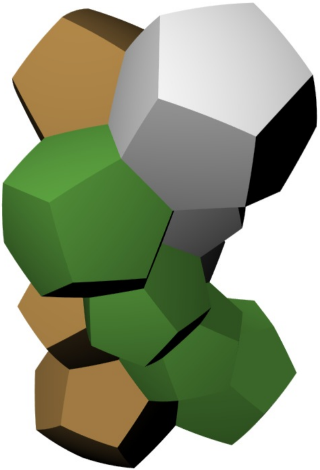

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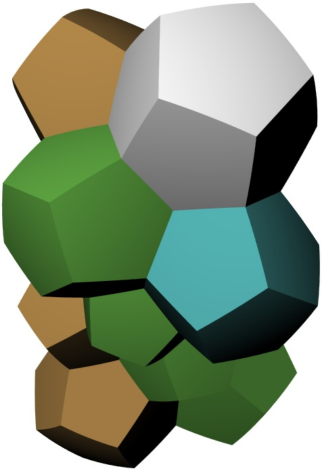

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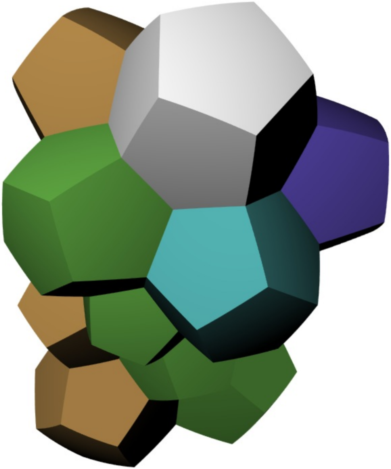

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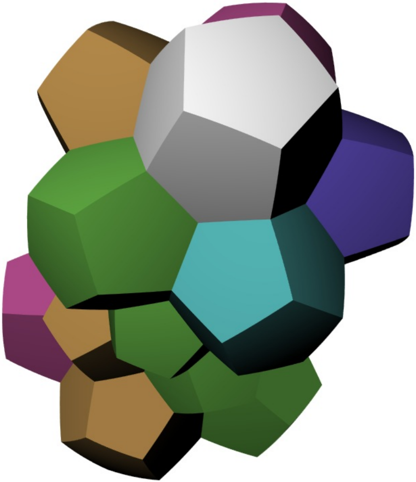

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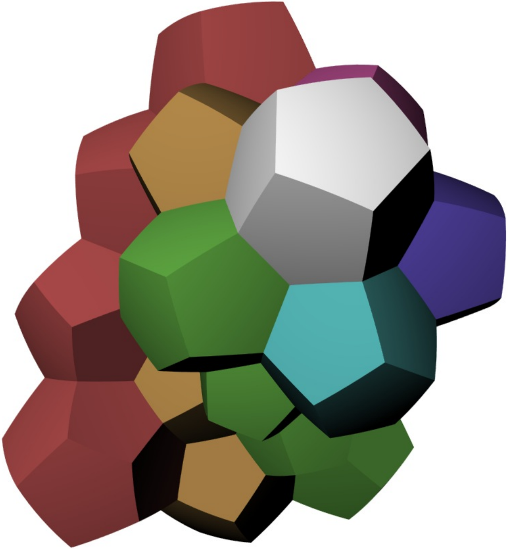

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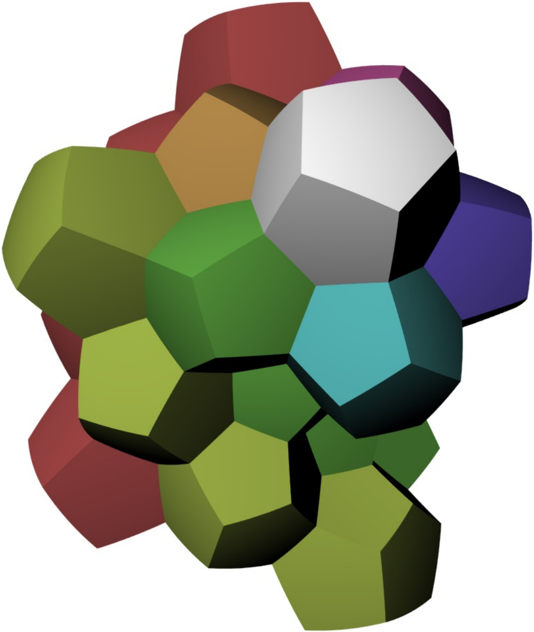

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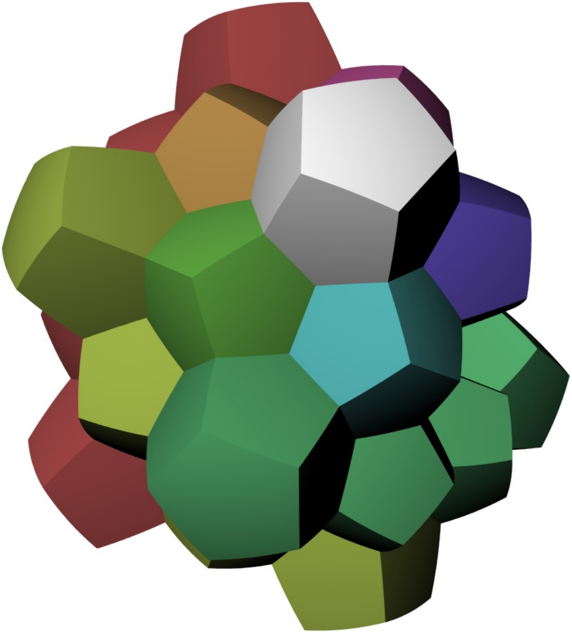

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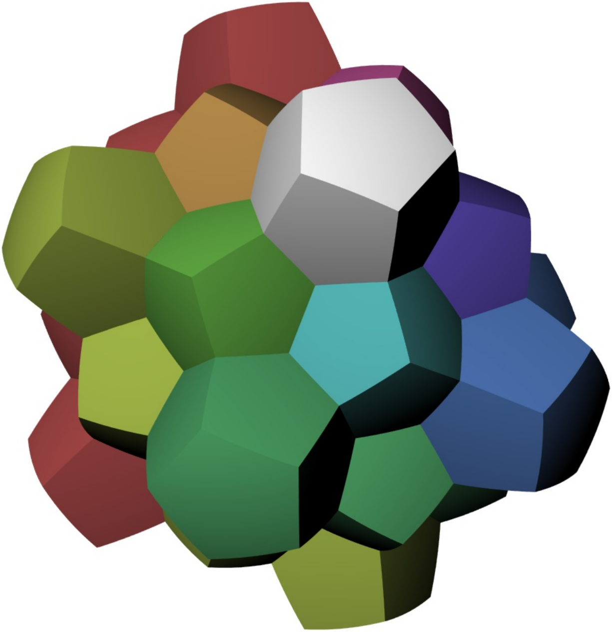

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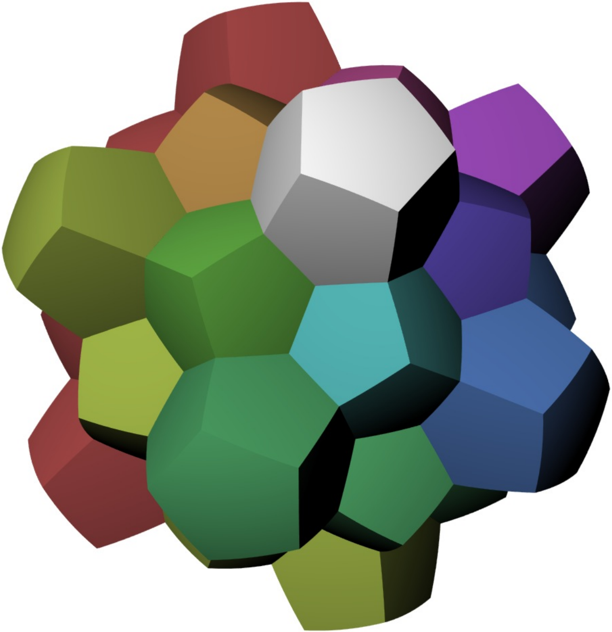
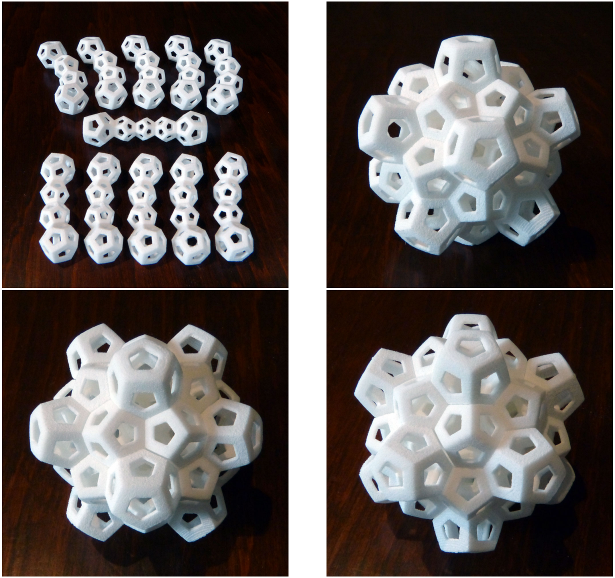
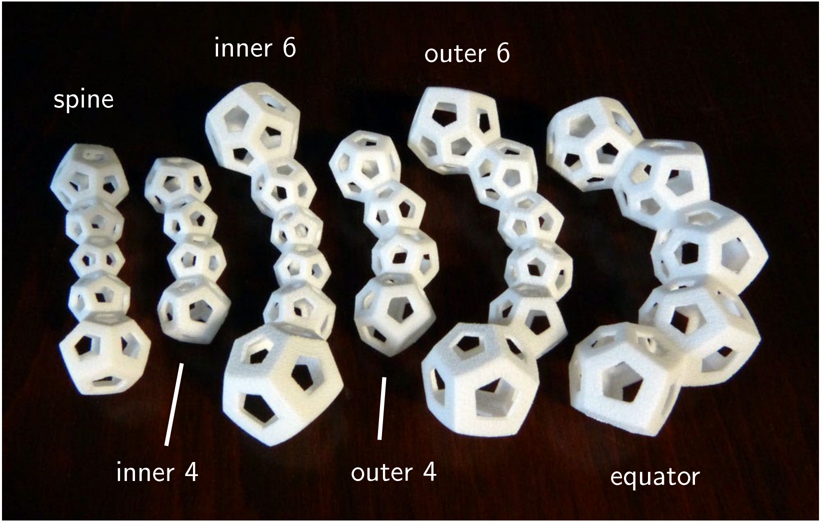


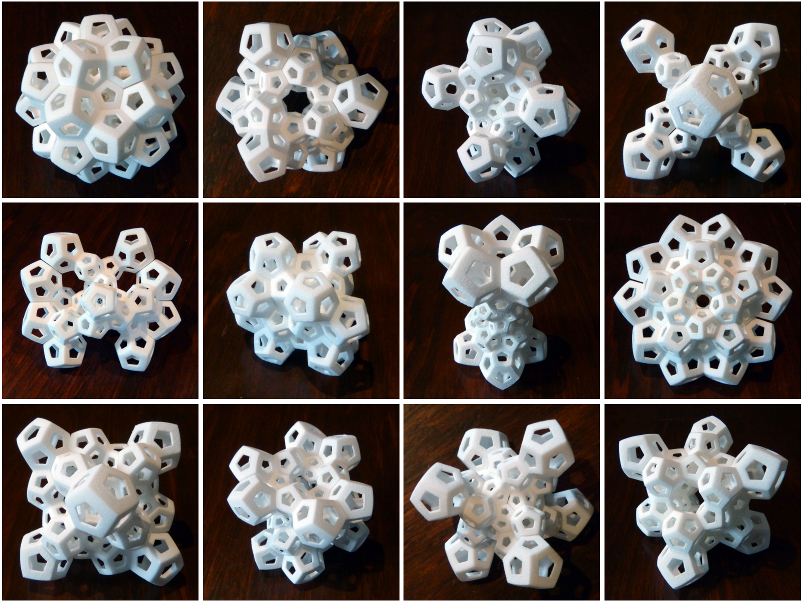
Dc45 Meteor puzzle



Six kinds of ribs



These make many puzzles, which we collectively call Quintessence.



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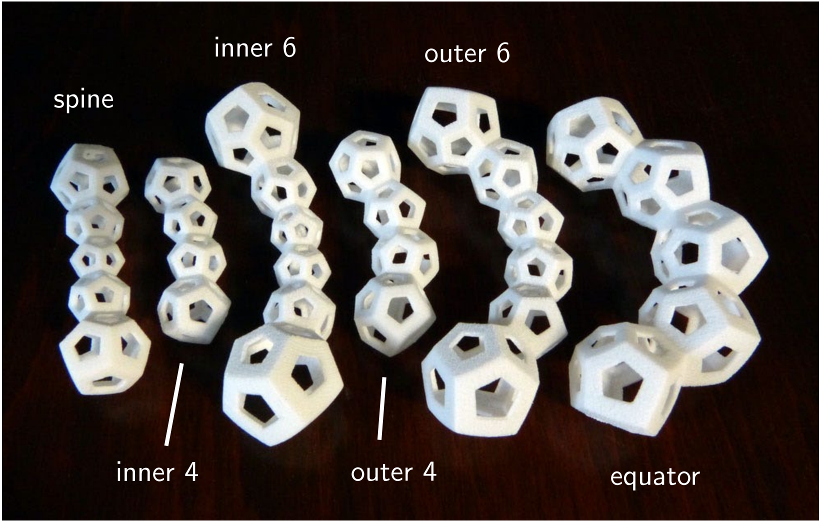

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- *At most six inner ribs are used in any puzzle.*
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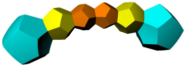

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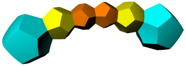

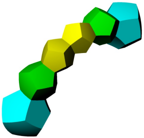

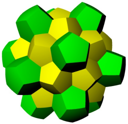

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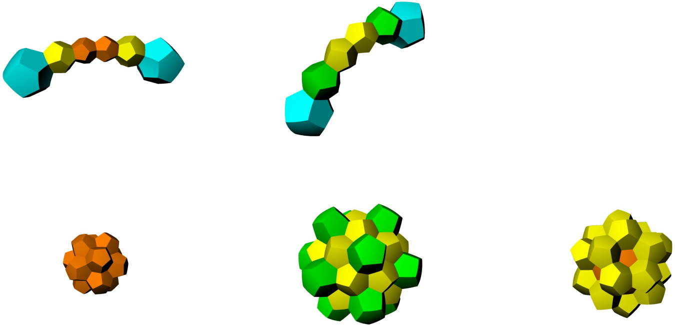

□

Links

https://www.ams.org/notices/201511/rnoti-p1309.pdf (Paper)
https://www.youtube.com/watch?v=c6U2_bwAcHM (Video)
https://homepages.warwick.ac.uk/~masgar (Webpage)
https://segerman.org (Webpage)