Computing in -- {low-dimensional topology

[Warwick 2017-11-6]

Goal: Explain ~~how~~ how computers are used via one problem.

<u>HOMEO</u>: Given M, N manifolds is M homeomorphic to N?

Lets restrict to closed, conn. oriented manifolds

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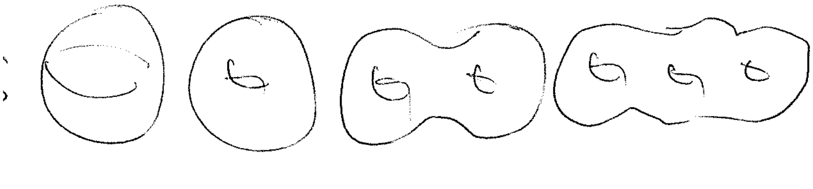
we discuss the problem in each dimension:

$\boxed{n=0}$ $M \cong N \cong \{pt\}$.

$\boxed{n=1}$ $M \cong N \cong S^1$

So far so good!

$\boxed{n=2}$ The classification of surfaces tells us that $\chi(S)$ is a complete invariant.



2 0 -2 -4

<u>Thm</u>: Suppose S admits a hyperbolic metric. Then $\text{Area}(S) = -2\pi \chi(S)$.

Area is multiplicative under covers (as is χ-char) so

<u>Suprise</u>: Any two connected d-fold covers of S are homeomorphic.

$\boxed{n \geq 4}$ All such problems are undecidable, hence the title of the talk.

$\boxed{n=3}$ The rest of the talk.

<u>Rmk</u>: If M is closed, conn, and odd dimensional then $\chi(M) = 0$. Sad!

<u>Geometrization</u> [Thurston, Perelman].

Suppose M is connected, closed, oriented ~~[illegible]~~ three-manifold. Then M has a (canonical) decomposition along spheres and tori into geometric pieces.

So we must also consider manifolds with torus boundary.

Example: Knot and link complements in S^3.

Examples

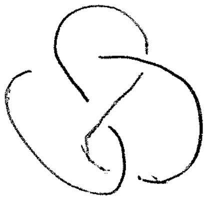
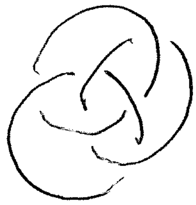


fig 8



Borromean Rings.

Form $S^3 - n(K)$, etc.

Mostow $[n \geq 3]$ If M^n, N^n are closed hyp n-mfds and $\pi_1(M) \cong \pi_1(N)$ then M is isometric to N.

Thus geometric invariants like volume are topological invariants.

[Said another way: hyp structures, when they exist are unique]

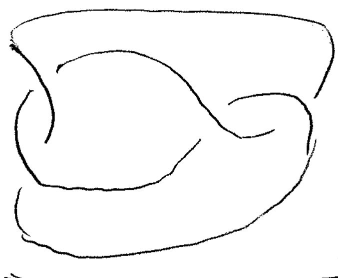
Snappy

Maintained by Culler-Dunfield, kernel code written by Weeks with input by many others.

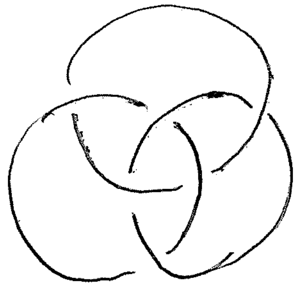
[Snappy Demo]

(1) M = Manifold()

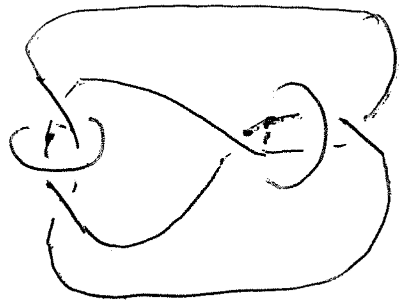
Draw figure 8 knot



(2) B = Manifold()



(3) A = Manifold()



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uses python under the hood so.

everything is object oriented.

Methods to try:

M.identify()
M.volume()
M.covers(4)

X, Y = M.covers(4)

X.identify(), Y.identify()
X.is_isometric_to(Y)
X.num_cusps()
Y.num_cusps()
A.is_isometric_to(B)

Computop: Dunfield maintains a webpage of topological software and other useful references

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KnotInfo [and others]

Questions:

(1) Does the computer disappear?

A) Sometimes: often it is nice to find a non-computer proof

(5) How are computations verified?

(A) A recent addition to Snappy uses interval arithmetic and ~~the~~ a rigorous version of Newton's method.

Let's try a simple example [in Sage]

M.verify_hyperbolicity()

OK: the numbers are verified shapes of tetrahedra. Lets try a larger example. [Snappy and Sage]

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(2) What is a big problem?

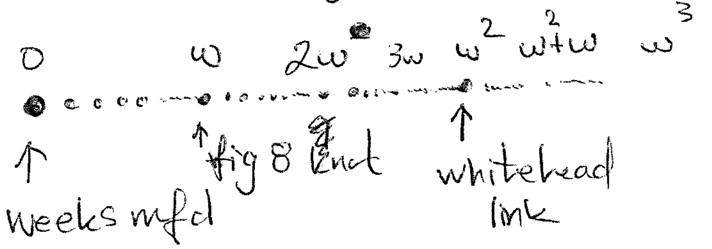
To verify hyperbolicity (if all tetrahedra are positive) I don't know!

Computing volumes will work up to hundreds of tetrahedra (thousands?)

Dirichlet domains will fail much sooner!

3 Databases? Are Great!

<u>Theorem</u> [Thurston Jørgenson]
The set of volumes of finite vol hyp 3-manifolds has order type ω^ω



The two oldest censuses are the Weeks census and the (extended) Rolfsen knot tables [orginally started by Tait!]