Sphere recognition is in **NP**

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Problem: S^3 RECOGNITION

Instance: A triangulation, T.

Question: Is $|T|$ homeomorphic to the three-sphere?

Theorem 1 (Rubinstein, Thompson). *The three-sphere recognition problem is decidable.*

Theorem 2 (S-). *The three-sphere recognition problem lies in* **NP***.*

We use:

- In any triangulation of S^3 there is an *almost normal two-sphere* [Rubinstein, Thompson].

- Casson's exponential time version of Thompson's algorithm.

- The *orbit counting* algorithm due to Agol, Hass, and Thurston. (There is a similar algorithm due to Schaefer, Sedgwick, and Stefankovic.)

Normal and almost normal: A properly embedded surface $S \subset |T|$ is *normal* if S intersects every tetrahedron in a collection of *normal disks*. We say S is almost normal if S is normal except for exactly one *almost normal piece*.

[Picture]

We will also need:

Theorem 3 (S-). *There is a polynomial time algorithm which, given a triangulation T and a transversely oriented almost normal surface $v(S)$, computes $v(\widetilde{S})$.*

Here $v(S)$ is the *coefficient vector* of normal disks. The surface $\widetilde{S}$ is the *normalization* of the almost normal surface S.

Certificate: A sequence $\{(T_i, v(S_i))\}_{i=1}^{m}$ where T_i is a triangulation and $v(S_i)$ is a surface vector.

Construction I: [following Casson] Suppose that T is a triangulation of S^3. Check that $|T|$ is a homology three-sphere. Let $T_1 = T$. Inductively we have:

- If T_i is not *zero-efficient* [Jaco-Rubinstein] then let $S_i \subset |T_i|$ be a fundamental non-trivial normal two-sphere. Let T_{i+1} be the triangulation obtained by *crushing* T_i along S_i.

- If T_i is zero-efficient and non-empty then let $S_i \subset |T'| \subset |T_i|$ be a fundamental almost normal two-sphere, if any exist. In this case let $T_{i+1} = T_i \setminus T'$.

[Continued]

A normal or almost normal surface S is *fundamental* if $v(S)$ is not a sum of two non-zero surface vectors.

Construction II:

- If T_i is zero-efficient, non-empty, and no fundamental almost normal two-sphere exists then halt: T was not a triangulation of S^3.

- If $T_i = \emptyset$ then take $m = i$ and $S_m = \emptyset$. In this case $|T|$ was the three-sphere.

Checking I:

- Check that $|T|$ is a homology three-sphere.
- Check that $T = T_1$.
- Check that $\chi(S_i) = 2$.
- Check that S_i is connected using [AHT].

[Continued]

There is a tricky point here – we can check the last two bullets in polynomial time *in the number of tetrahedra of* T_i. This is because the S_i are fundamental, implying that the entries in $v(S_i)$ have at most polynomially many digits.

Checking II:

- If S_i is normal then crush T_i along S_i to obtain T_i' [Jaco-Rubinstein]. Check that $T_{i+1} = T_i'$.

- If $S_i \subset |T'| \subset |T_i|$ is almost normal then normalize S_i (in both directions) to obtain $\widetilde{S_i}^+$ and $\widetilde{S_i}^-$ [Theorem 3]. Check that $\widetilde{S_i}^+ \cup \widetilde{S_i}^-$ is a union of vertex links. Check that $T_{i+1} = T_i \setminus T'$.

- Check that T_m is the empty set.

Another tricky point – for the second bullet we don't check that T_i is zero-efficent. Instead normalize S_i in polynomial time (in the number of tetrahedra of T_i).

Proof of Theorem 3:

- Use the "thick-thin" decomposition of $|T| \setminus S$. The components of the thin region (*i.e.*, I-bundle regions) are computed using [AHT].

- Understand Haken's normalization procedure, as applied to a transversely oriented almost normal surface. [Jaco-Rubinstein, S-].

[Picture]

For details please consult `math.GT/0407047` at the ArXiV.