Thurston theory for critically fixed branched covering maps

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Branched covers

Definition.

Continuous map $f\colon S^2 \to S^2$ is called a branched covering map if there exist two finite sets $A, B \subset S^2$ such that $f\colon S^2 \backslash B \to S^2 \backslash A$ is a covering map.

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Any rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

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C_f is a set of all critical points of the map f

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Orientation-preserving branched covering map $f \colon S^2 \to S^2$ is called a Thurston map if f is postcritically finite (pcf), i.e., P_f is finite.

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   - $$f(z) = z^2 - 1$$ $$C_f = \{0, \infty\}, P_f = \{0, -1, \infty\}$$
   - $$f(z) = \frac{3z^5 - 20z}{5z^4 - 12}$$ $$C_f = P_f = \{\pm 1 \pm i\}$$

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2. $\varphi \circ f \circ \psi$, where f is a pcf rational map, $\varphi, \psi \in \mathrm{Homeo}^+(S^2)$ such that $\varphi(P_f) = \psi(P_f) = P_f$.

Blow-up operation [Kevin Pilgrim-Tan Lei'92]

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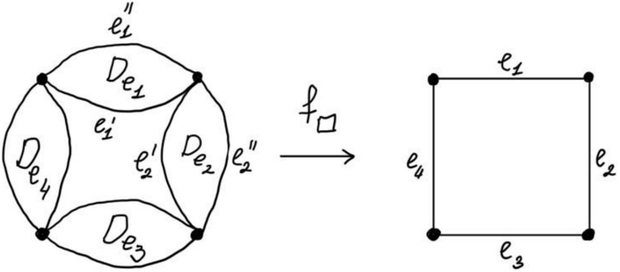

$$\mathrm{Int}(D_{e_i}) \xrightarrow[\text{homeo}]{f_\square} S^2 \backslash e_i,$$

$$e_i', e_i'' \xrightarrow[\text{homeo}]{f_\square} e_i,$$

$$S^2 \backslash \bigcup D_{e_i} \xrightarrow[\text{homeo}]{f_\square} S^2 \backslash \bigcup e_i.$$

Isotopy

Definition.

Let f and g be two Thurston maps with the same postcritical set P. We say that f is isotopic to g if

- $f = g \circ \varphi$ and $\varphi \in \mathrm{Homeo}^+(S^2)$;
- φ is homotopic rel. P to id_{S^2}.

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Example.

Result of blowing up a planar connected graph is defined uniquely up to isotopy of Thurston maps.

Mapping Class Group

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$$\mathrm{BrMod}(S^2, A) = \{f \in \mathrm{Homeo}^+(S^2),\ \text{where } f \text{ is a Thurston map},\ P_f \subset A,\ \text{and } f(A) \subset A\}/\sim$$

Combinatorial equivalence

Definition.

Two Thurston maps f and g are combinatorially equivalence (or Thurston equivalent) if there exist two other Thurston maps $\widetilde{f}$ and $\widetilde{g}$ such that

- f and $\widetilde{f}$ are isotopic,
- g and $\widetilde{g}$ are isotopic,
- $\widetilde{f}$ and $\widetilde{g}$ are (topologically) conjugate.

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Theorem (W. Thurston'80s; Douady-Hubbard'93).

Thurston map[a] f is realized if and only if f have no Thurston obstruction.

[a] with hyperbolic orbifold

Obstructions

Let $f\colon S^2 \to S^2$ be a Thurston map with a postsingular set P_f.

Definition.

Simple closed essential curve $\gamma \subset S^2 \backslash P_f$ is called a Levy fixed curve for f if there exists a simple closed curve $\gamma' \subset f^{-1}(\gamma)$ such that

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Theorem (Hlushchanka-NP'23).

Let f be a critically fixed Thurston map. Then f is realized if and only if it has no Levy fixed curves.

Classification of critically fixed rational maps

Theorem (Hlushchanka'19, Pilgrim et al'14).

Critically fixed rational maps (up to conjugation by Möbius maps) $\xleftrightarrow{1:1}$ Planar connected graphs on S^2 with parallel edges, without loops (up to isomorphism)

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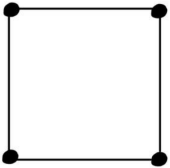
$$f(z) = \frac{3z^5 - 20z}{5z^4 - 12} \longleftrightarrow$$



Charge(f)

The case of a non-connected graph

Definition.

Let

- $G = (V, E)$ be a planar graph on S^2 without isolated points,
- $\varphi \in \mathrm{Homeo}^+(S^2)$ such that $\varphi|G = \mathrm{id}_G$.

In this case, we say that (G, φ) is an admissible pair.

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Theorem (Hlushchanka - NP'23).

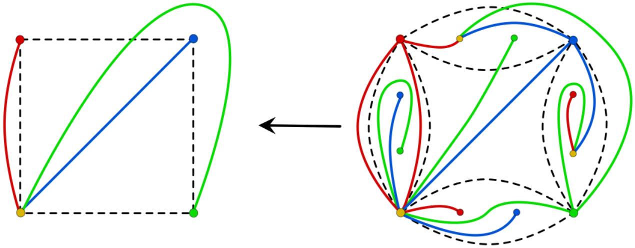
Critically fixed Thurston maps (up to an isotopy) $\xleftrightarrow{1:1}$ Admissible pairs (G, φ) (up to homotopy rel. $V(G)$)

Classification problem

Example.

- pcf topological polynomials [Poirier'10; Belk-Lanier-Margalit-Winarski'21]
- pcf Newton maps [Drach-Lodge-Mikulich-Schleicher'21]

Trees and critically fixed Thurston maps



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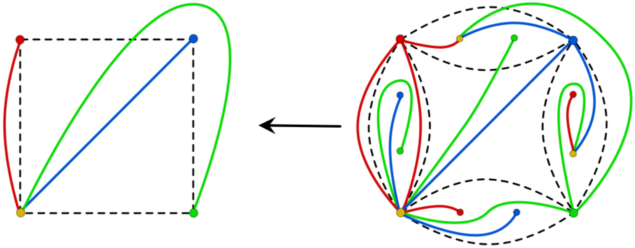

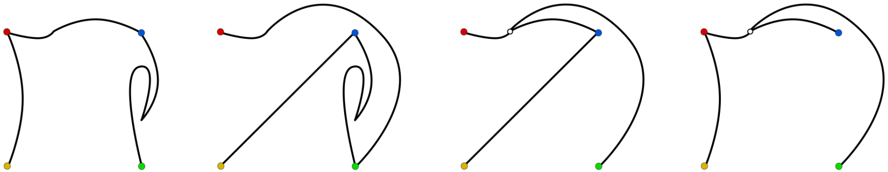

Sequence of pullbacks of trees

Theorem (Hlushchanka-NP'23).

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- f be a critically fixed Thurston theory,
- $\{T_n\}_{n\geqslant 0}$ sequence of trees such that $T_{n+1} \in \Pi_f(T_n)$ for all $n \geqslant 0$.

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Remark.

The set $\mathcal{N}_f$ is finite if and only if the graph $\mathrm{Charge}(f)$ is connected (i.e., the map f is realized).

Controlling pullbacks

Proposition (Hlushchanka-NP'23).

Let f be a critically fixed Thurston map and T be a planar embedded tree. Then for each edge $e \in \mathrm{Charge}(f)$ we have:

1. $i_{C_f}(f^{-1}(T), e) \leqslant i_{C_f}(T, e)$;

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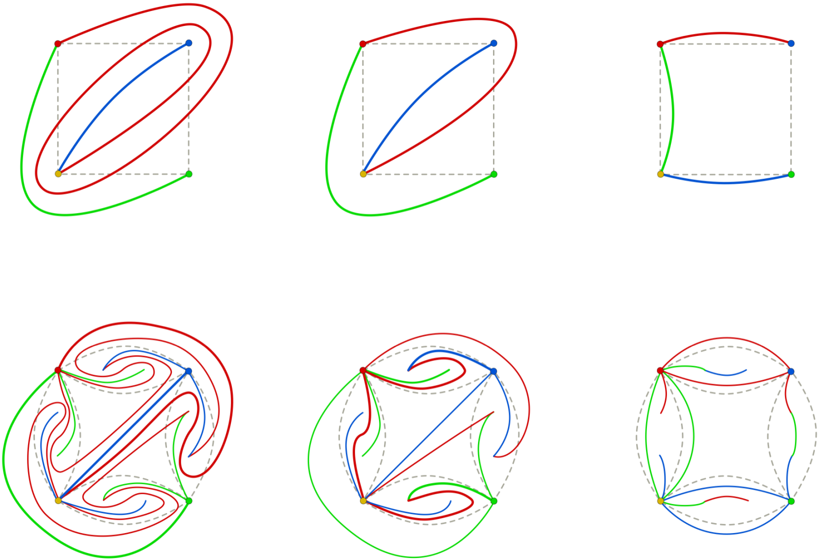
2. $i_{C_f}(f^{-1}(T), e) < i_{C_f}(T, e)$, if $i_{C_f}(T, e) > 0$.

Algorithm



Thank you for your attention !