Dynamical Approximation of Entire Functions

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Entire functions

An entire function is a holomorphic function $f : \mathbb{C} \to \mathbb{C}$.
Example: e^z, $\cos z$, $\int_0^z e^{-w^2} dw$.

Definition 1

*Given f entire, $a \in \mathbb{C}$ is called an **asymptotic value** for f if there exists an arc $\gamma : [0, \infty) \to \mathbb{C}$ such that $\gamma(t) \to \infty$ and $f(\gamma(t)) \to a$ as $t \to \infty$.*

Example: 0 is an asymptotic value for e^z, and $\frac{\sqrt{\pi}}{2}$ and $-\frac{\sqrt{\pi}}{2}$ are asymptotic values for $\int_0^z e^{-w^2} dw$. $\cos z$ has no asymptotic values.

Picard's Theorem: In any neighborhood U of ∞, for every point $w \in \mathbb{C}$ except at most one, there exist infinitely many points $z \in U$ satisfying $f(z) = w$.

Therefore, an entire function can omit at most one point on the plane. Asymptotic values need not be omitted values. For example, $\int_0^z e^{-w^2} dw$ is surjective.

Postsingular finiteness

Fix f entire.

Definition 2

*Any critical or asymptotic value of f is called a **singular value**. The closure of the set of all singular values is denoted S_f.*

Singular values are defined analogously for any branched cover $f : \mathbb{R}^2 \to \mathbb{R}^2$. S_f could be finite or infinite.

Definition 3

Given a branched cover $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, the postsingular set of f is defined as

$$P_f = \bigcup_{n \geq 0} f^{\circ n}(S_f)$$

f is called postsingularly finite (PSF) if P_f is finite.

Example: $f(z) = 2\pi i e^z$.
For polynomials, we use the terminology "postcritically" finite (PCF) instead of postsingularly finite, since there are no asymptotic values.

Post singular finiteness

Definition 4

A PSF map $f : \mathbb{R}^2 \to \mathbb{R}^2$ *is a topological polynomial if it has finite degree.*

Definition 5

A PSF map $f : (\mathbb{R}^2, P_f) \to (\mathbb{R}^2, P_f)$ *is topologically entire if for every orientation preserving quasiconformal map* $\varphi : \mathbb{R}^2 \to \mathbb{C}$*, there exists an orientation preserving quasiconformal* $\psi : \mathbb{R}^2 \to \mathbb{C}$ *such that* $g = \varphi \circ f \circ \psi^{-1} : \mathbb{C} \to \mathbb{C}$ *is entire.*

The above is not true for all PSF maps; g could look like a holomorphic map from $\mathbb{D}$ to $\mathbb{C}$.

Complex dynamics

For any entire function f, the Fatou set $\mathcal{F}_f$ is defined as the set of $z \in \mathbb{C}$ such that the family of functions $f^{\circ n}$ is normal in some neighborhood of z. The Julia set $J_f = \mathbb{C} \setminus \mathcal{F}_f$. The escaping set $\mathcal{I}_f$ is the collection of z such that $f^{\circ n}(z) \to \infty$ as $n \to \infty$.

PSF functions have nice geometrical properties. If $f : \mathbb{C} \to \mathbb{C}$ is PSF entire and does not extend holomorphically over ∞,

1. $\mathcal{F}_f = \bigcup_{w \text{ s.a.}} \mathcal{A}(f, w)$
2. $J_f = \overline{\mathcal{I}_f}$

Example: If $f(z) = \lambda e^z$ is PSF, since it has no critical points, $J_f = \mathbb{C}$.

If f is a PCF polynomial,

1. $\mathcal{F}_f = \mathcal{I}_f \cup \bigcup_{w \text{ s.a.}} \mathcal{A}(f, w)$
2. $J_f = \partial \mathcal{I}_f$

It has hard to come up with holomorphic PSF maps f with prescribed dynamics on P_f, so we work with topological models instead. We will later talk about how we can produce holomorphic maps from topological ones.

Dynamical approximations

Broadly speaking, given an entire function f, and a sequence of entire functions f_n, we say that $f_n \to f$ **dynamically** if $f_n \to f$ locally uniformly on $\mathbb{C}$, and there is a certain dynamical property X of f such that f_n satisfy X for all n.

In practice, if we can find such a sequence of functions f_n that are simpler to study than f is, in principle, we can study the property X on f by looking at the f_n's instead. Often, we want f_n's to be polynomials.

The property X could be anything.

Question 1: If f is hyperbolic(parabolic), can we find polynomials f_n hyperbolic(resp. parabolic) such that $f_n \to f$ locally uniformly on $\mathbb{C}$?

Question 2: If f is PSF, can we find PCF polynomials f_n such that $f_n \to f$ locally uniformly on $\mathbb{C}$?

Example: Consider the exponential function $f(z) = \frac{1}{e}e^z$. $z = 1$ is a fixed point for this function, and has multiplier 1. Therefore the function is parabolic.
Given $f_n(z) = (1 - \frac{1}{n})^{n-1}(1 + \frac{z}{n})^n$, we note that $z = \frac{1}{1-\frac{1}{n}}$ is again a fixed point of f_n with multiplier 1, and that $f_n \to f$ locally uniformly on $\mathbb{C}$.

Results

- Devaney, Lyubich, Rempe, Schleicher and others: Broad studies of several transcendental maps as limits of rational maps, or by generalizing techniques used for rational maps
- Kisaka ([Kis95]): If $f_n \to f$ locally uniformly in $\mathbb{C}$ and $\mathcal{F}_f = \cup_{w \text{ s.a.}} \mathcal{A}(f, w)$, then $J_{f_n} \to J_f$ in the Hausdorff metric.
- Bodelón-Devaney-Hayes-Roberts-Goldberg-Hubbard([BDH$^+$00]): Escaping parameters and hyperbolic components of exponentials as a limit of unicritical maps
- Mihaljević-Brandt ([MB11]): Kernel convergence of nonescaping-hyperbolic components in the general entire setting

Theorem 6 (M.)

Given a PSF exponential $f(z) = \lambda e^z$, there exist PCF polynomials $f_n(z) = \lambda_n(1 + \frac{z}{n})^n$ that converge to f locally uniformly, such that $f_n|P_{f_n}$ is conjugate to $f|P_f$ for all n. There exists a polynomial $Q = Q(f) \in \mathbb{Z}[X]$ and integers $\ell, k \geq 1$ such that the parameter ray at angle $\frac{Q(n)}{n^\ell(n^k - 1)}$ lands at f_n for each n.

Example: For $f(z) = 2\pi i e^z$, $f_n(z) = n(e^{\frac{2\pi i}{n}} - 1)(1 + \frac{z}{n})^n$.

Theorem 7 (M., Prochorov, Reinke)

Given a PSF entire function f, there exist PCF polynomials f_n such that $f_n \to f$ locally uniformly on $\mathbb{C}$ and $f_n|P_{f_n}$ is conjugate to $f|P_f$ for all n.

Thurston equivalence: Topological model $\to$ holomorphic map

Definition 8

Two topological entire PSF maps f and g are said to be ***Thurston equivalent*** *if there exist orientation preserving homeomorphisms $\varphi_0, \varphi_1 : \mathbb{R}^2 \to \mathbb{R}^2$ that are isotopic relative to P_f and the following diagram commutes:*

$$\begin{array}{ccc} (\mathbb{R}^2, P_f) & \xrightarrow{\varphi_1} & (\mathbb{R}^2, P_g) \\ \downarrow f & & \downarrow g \\ (\mathbb{R}^2, P_f) & \xrightarrow{\varphi_0} & (\mathbb{R}^2, P_g) \end{array}$$

We say that f ***is realized*** *if there exists a PSF entire function g Thurston equivalent to f, and* ***obstructed*** *otherwise.*

If f is a topological polynomial, it is obstructed if and only if it has a topological obstruction called a *Levy cycle* ([DH93], Levy-Bernstein Theorem [Hub16]). This has also been extended to special families of entire functions such as the exponentials ([HSS09]) and the structurally finite family (Shemyakov). It is not known whether this is true for all PSF maps.

The Thurston pullback operator

Let g be a PSF topological entire function.

Definition 9

The Teichmüller space $\text{Teich}(\mathbb{R}^2, P_g)$ *is defined as the collection of orientation preserving homeomorphisms* $\varphi : \mathbb{R}^2 \to \mathbb{C}$ *modulo an equivalence relation* $\sim$ *where* $\varphi \sim \psi$ *if there exists an affine map* $M : \mathbb{C} \to \mathbb{C}$ *such that* $M \circ \varphi$ *is isotopic to* ψ *relative to* P_g*.*
Associated to g *is the* ***Thurston pullback operator*** σ_g *with* $\sigma_g([\varphi]) = [\psi]$ *where* ψ *satisfies*

$$\begin{array}{ccc} (\mathbb{R}^2, P_g) & \xrightarrow{\psi} & (\mathbb{C}, \psi(P_g)) \\ \downarrow g & & \downarrow h \\ (\mathbb{R}^2, P_g) & \xrightarrow{\varphi} & (\mathbb{C}, \varphi(P_g)) \end{array}$$

with $h = \varphi \circ g \circ \psi^{-1}$ *entire.*

- σ_g has a fixed point $[\varphi]$ if and only if g is realized.
- σ_g is weakly contracting in the Teichmüller metric, so a fixed point for σ_g, if it exists, is unique.

Strategy for proof of theorem 7

We first show:

Theorem 10 (M., Prochorov, Reinke)

Given any topological PSF entire map g, there exists a sequence of PCF topological polynomials g_n such that

- $P_{g_n} = P_g$
- *for every compact set* $K \subset \mathbb{R}^2$, $g_n|K = g|K$ *for large enough* n
- $\sigma_{g_n} \to \sigma_g$ *locally uniformly on* $\mathit{Teich}(\mathbb{R}^2, P_g)$

Remark: The maps g_n satisfy $(g_n)_* \pi_1(\mathbb{R}^2 \setminus g_n^{-1}(P_g), \dagger_n) \to g_* \pi_1(\mathbb{R}^2 \setminus g^{-1}(P_g), \dagger)$, for some points $\dagger_n, \dagger$, $t \notin P_g$ with $g(\dagger) = g_n(\dagger_n) = t$. The above groups are subgroups of $\pi_1(\mathbb{R}^2 \setminus P_g, t)$.

1. If g is realized (i.e., σ_g has a fixed point $[\varphi]$), then for large n, σ_{g_n}'s all have fixed points $[\varphi_n]$, which tend to the fixed point of σ_g as $n \to \infty$
2. If our PSF map $f = \phi \circ g \circ \psi^{-1}$, we show that there exist homeomorphisms $\psi_n \sim \phi_n$ such that $f_n = \phi_n \circ g_n \circ \psi_n^{-1}$ is PCF, and $f_n \to f$ locally uniformly on $\mathbb{C}$.

Questions

- The deformation space $\mathscr{X}$ of f is the set of entire maps g quasiconformally equivalent to g. Do there exist polynomials f_n with deformation spaces $\mathscr{X}_n$ such that for all PSF $g \in \mathscr{X}$, there exist PCF $g_n \in \mathscr{X}_n$ such that $g_n \to g$ locally uniformly on $\mathbb{C}$?
- Say we start with a PCF topological entire function f instead, by Theorem 10, we can still produce topological polynomials f_n with $\sigma_{f_n} \to \sigma_f$. If f is realised, then it is easy to show that the f_n's are realised. Is the converse true? That is, if infinitely many σ_{f_n}'s have fixed points, does σ_f have a fixed point?
- Can we do the same for transcendental meromorphic functions?

Approximations at the parameter level

Figure: The non-escaping locus in the moduli space of unicritical polynomials with degree $2, 3$ and 4 respectively

Approximations at the parameter level

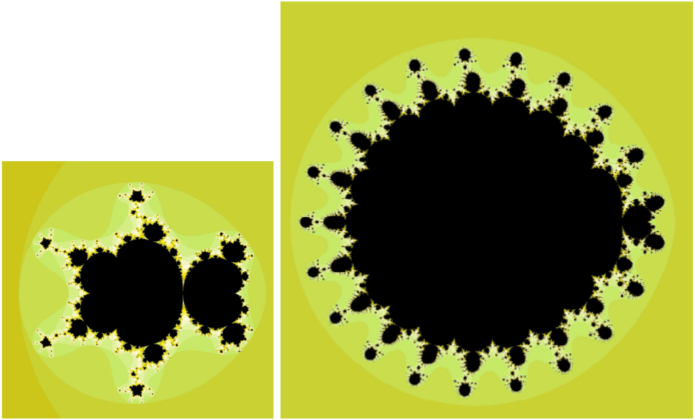

Figure: The non-escaping locus in the moduli space of unicritical polynomials with degree 5 and 20 respectively

Approximations at the parameter level

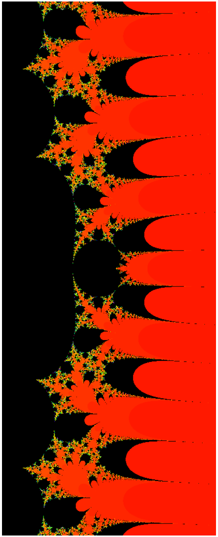

Figure: The non-escaping locus in the moduli space of exponential functions

References

C. Bodelón, R. L. Devaney, M. Hayes, G. Roberts, L. R. Goldberg & J. H. Hubbard – "Dynamical convergence of polynomials to the exponential", *J. Differ. Equations Appl.* **6** (2000), no. 3, p. 275–307.

A. Douady & J. H. Hubbard – "A proof of Thurston's topological characterization of rational functions", *Acta Math.* **171** (1993), no. 2, p. 263–297.

J. Hubbard, D. Schleicher & M. Shishikura – "Exponential Thurston maps and limits of quadratic differentials", *J. Amer. Math. Soc.* **22** (2009), no. 1, p. 77–117.

J. H. Hubbard – *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 2*, Matrix Editions, Ithaca, NY, 2016, Surface homeomorphisms and rational functions.

M. Kisaka – "Local uniform convergence and convergence of Julia sets", *Nonlinearity* **8** (1995), no. 2, p. 273–281.

H. MIHALJEVIĆ -BRANDT – "Dynamical approximation and kernels of non-escaping hyperbolic components", *Ergodic Theory and Dynamical Systems* **32** (2011), no. 4, p. 1418–1434.

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Thank you!