SPECIAL CUBULATION OF STRICT HYPERBOLIZATION

Lorenzo Ruffoni (Tufts)

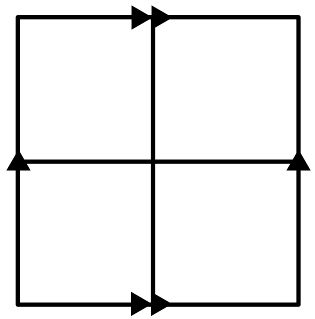
(joint work with J.-F. Lafont)

Geometry and Topology Seminar

Warwick - June 23, 2022

Special cubulation of strict hyperbolization

An example:



X

flat

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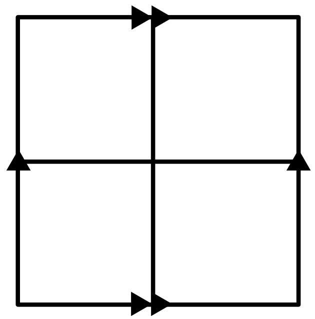

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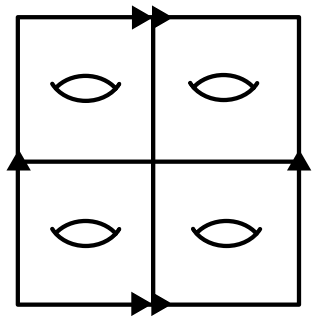

X_Γ

hyperbolic

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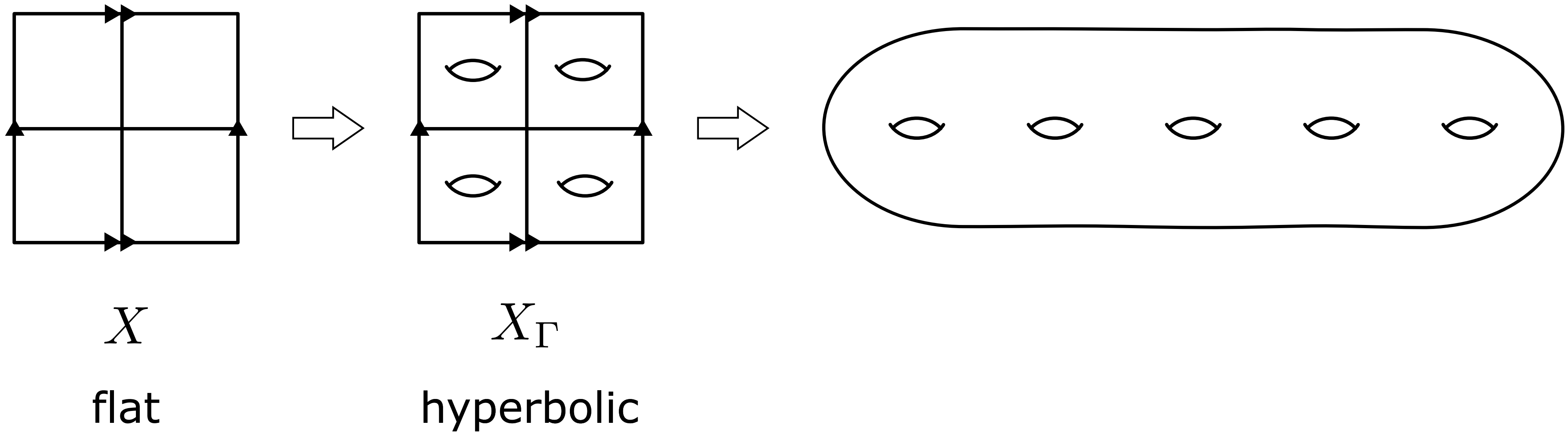

Special cubulation of strict hyperbolization

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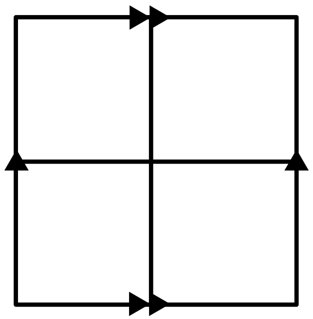

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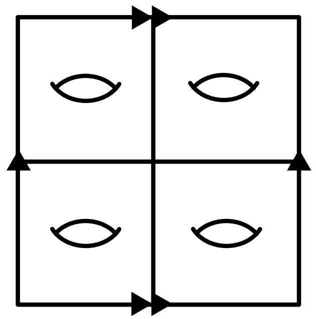
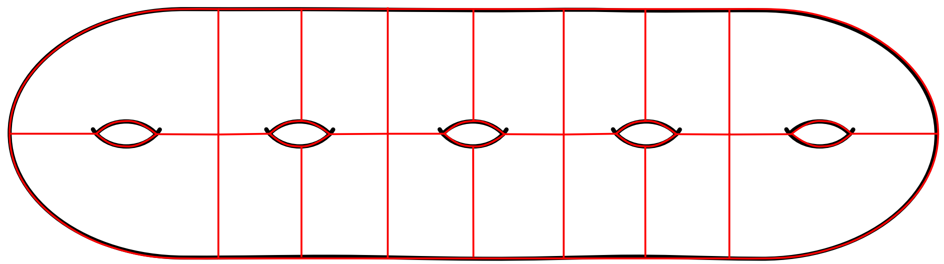


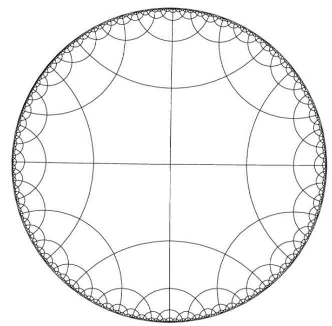
X_Γ

hyperbolic



$\pi_1(X_\Gamma) \hookrightarrow \mathrm{RACG}$

hyperbolic + res. finite



Special cubulation of strict hyperbolization

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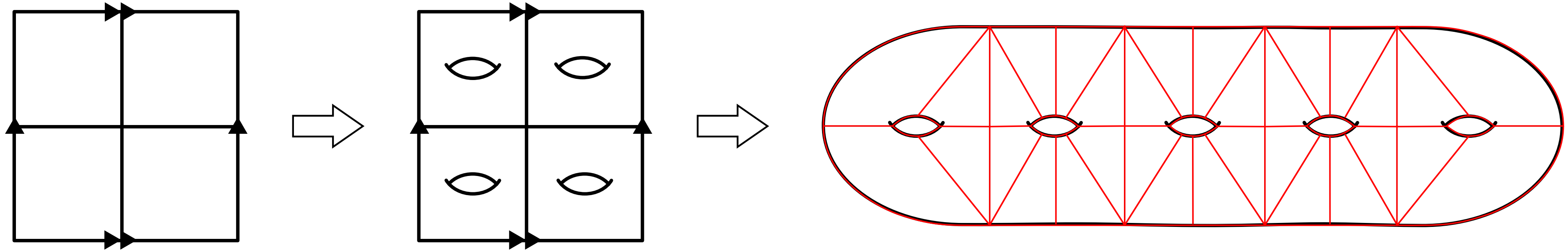
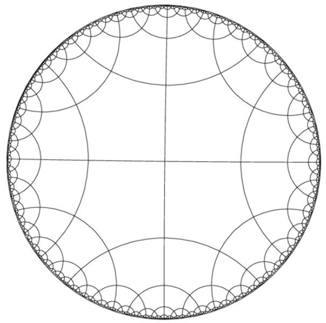


X	X_Γ	$\pi_1(X_\Gamma) \hookrightarrow \mathrm{RAAG}$
flat	hyperbolic	hyperbolic + res. finite



Special cubulation of strict hyperbolization

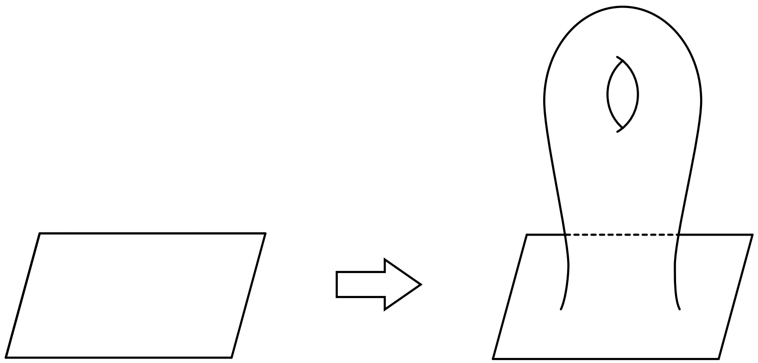
Theorem (Lafont, R., 2022)

Let X be a non-pos. curved, compact, foldable cubical complex. Then:

$$\pi_1(X_\Gamma) \curvearrowright \mathcal{C}(\widetilde{X_\Gamma})$$

cocompactly
isometrically

CAT(0) cubical complex



X $\qquad$ X_Γ

Charney-Davis
strict hyperbolization

Special cubulation of strict hyperbolization

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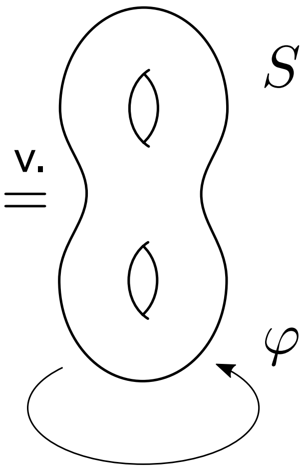
CAT(0) cubical complex

$\pi_1(X_\Gamma)$ hyperbolic + virtually special, and:

- virtually embed in RAAG, RACG
- RF, QCERF, RFRS
- some virtually algebraically fiber
- some are "new" (not rank 1 lattices)

$M \overset{\text{v.}}{=}$



Context: virtual fibering conjecture for hyperbolic 3-manifolds
(Thurston ... Kahn-Markovic, Sageev, Haglund-Wise, Agol, ...)

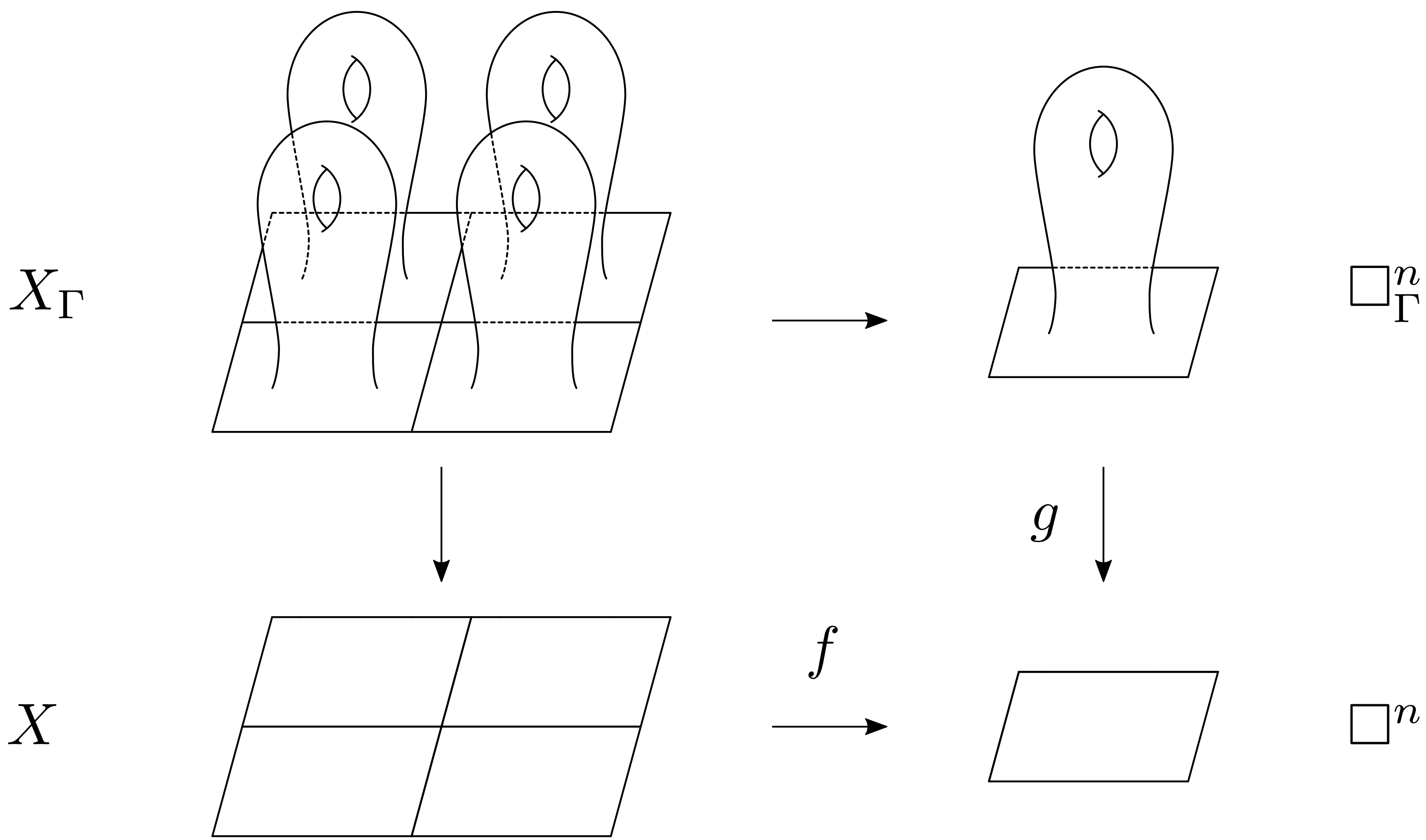
Special cubulation of **strict hyperbolization**

What: a procedure that

- makes spaces negatively curved
- preserves some topology

Why: construct **wild** examples of

- Gromov hyperbolic groups
- negatively curved manifolds

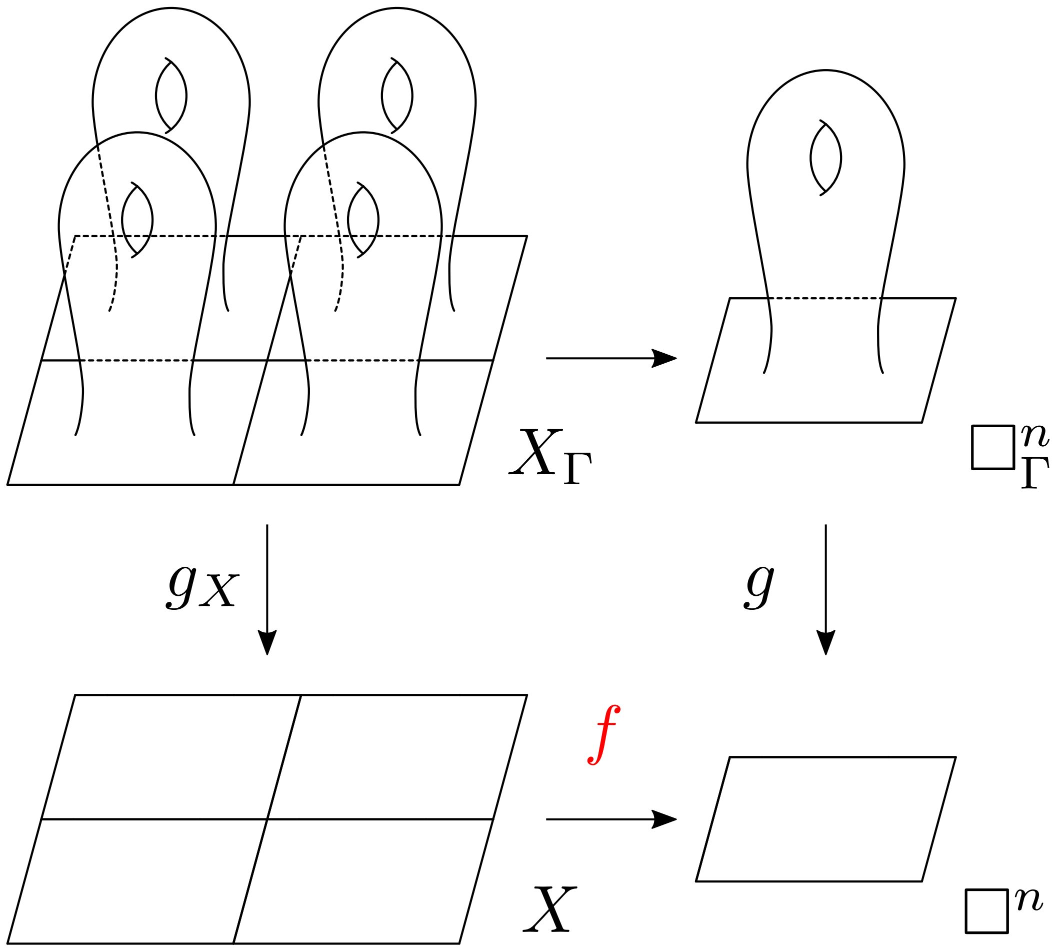


Special cubulation of **strict hyperbolization**

Theorem (Charney, Davis, 1995)

Let $X \xrightarrow{f} \square^n$ be a non-pos. curved, compact, foldable cubical complex.
Let $\Gamma \subseteq \mathrm{SO}_0(n,1)$ be a suitable arithmetic lattice. Then $\exists\, X_\Gamma$ s.t.

- X_Γ is a locally CAT(-1) and piecewise hyperbolic complex
- links in X = links in X_Γ
- g_X is injective in cohomology and surjective in homology



Applications

- Every triangulable manifold is cobordant to a neg.curved one
- Neg.curved manifolds/groups with ideal boundary $\neq S^n$

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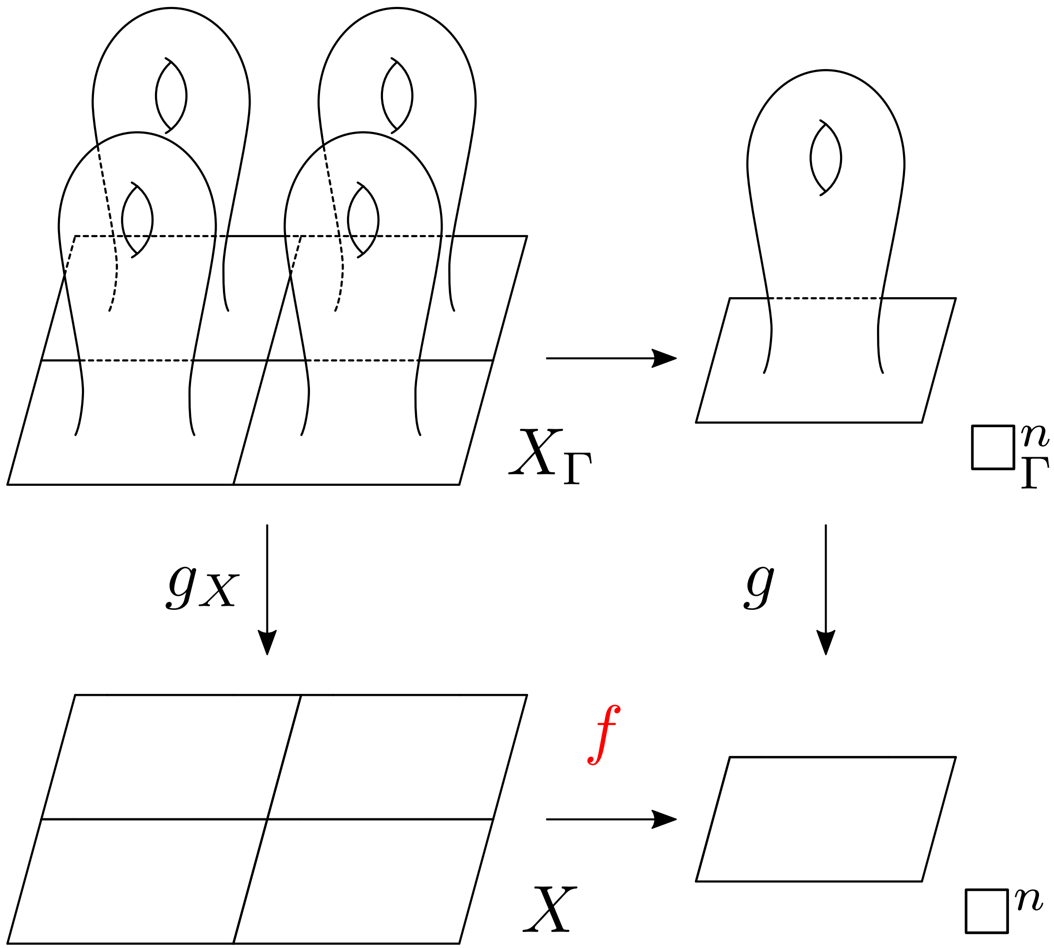


Applications **+ linear, res. finite**

- Every triangulable manifold is cobordant to a neg.curved one
- Neg.curved manifolds/groups with ideal boundary $\neq S^n$

Special cubulation of strict hyperbolization

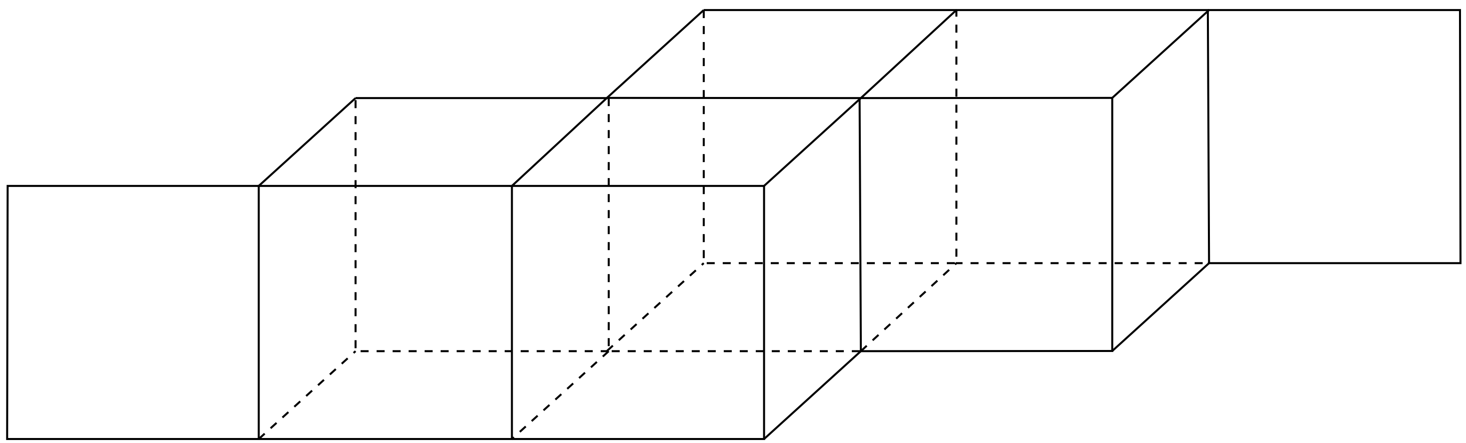
What: a procedure that

- makes groups act on cube compl.
- embeds groups in RAAGs

Why: prove a group is **nice**

- linear over the integers
- residual properties (RF, RFRS)

$G \curvearrowright$



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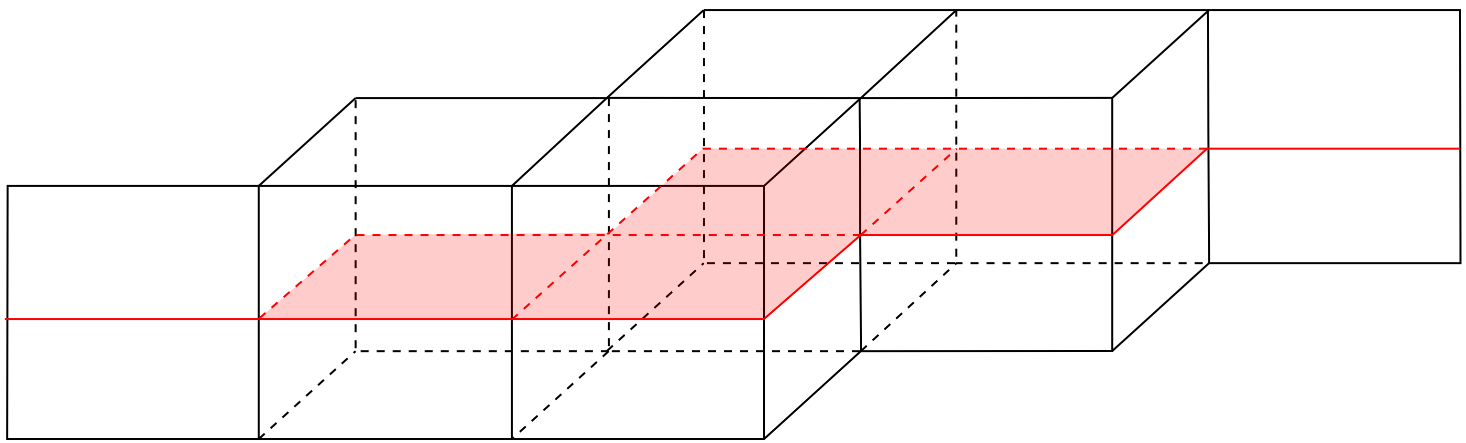
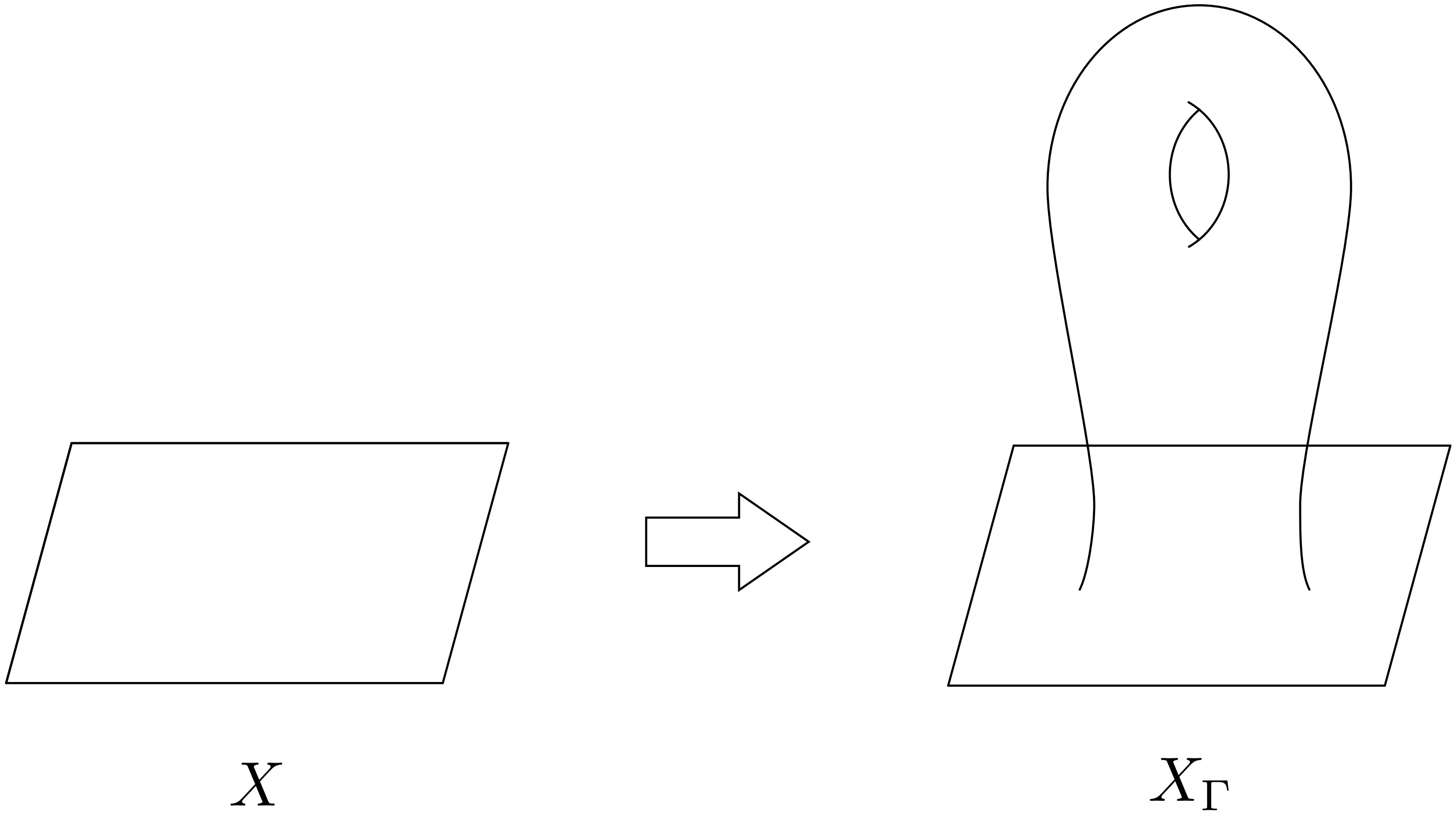


Common strategy: find many codimension-1 "convex" subgroups/subspaces

A problem: hyperplanes



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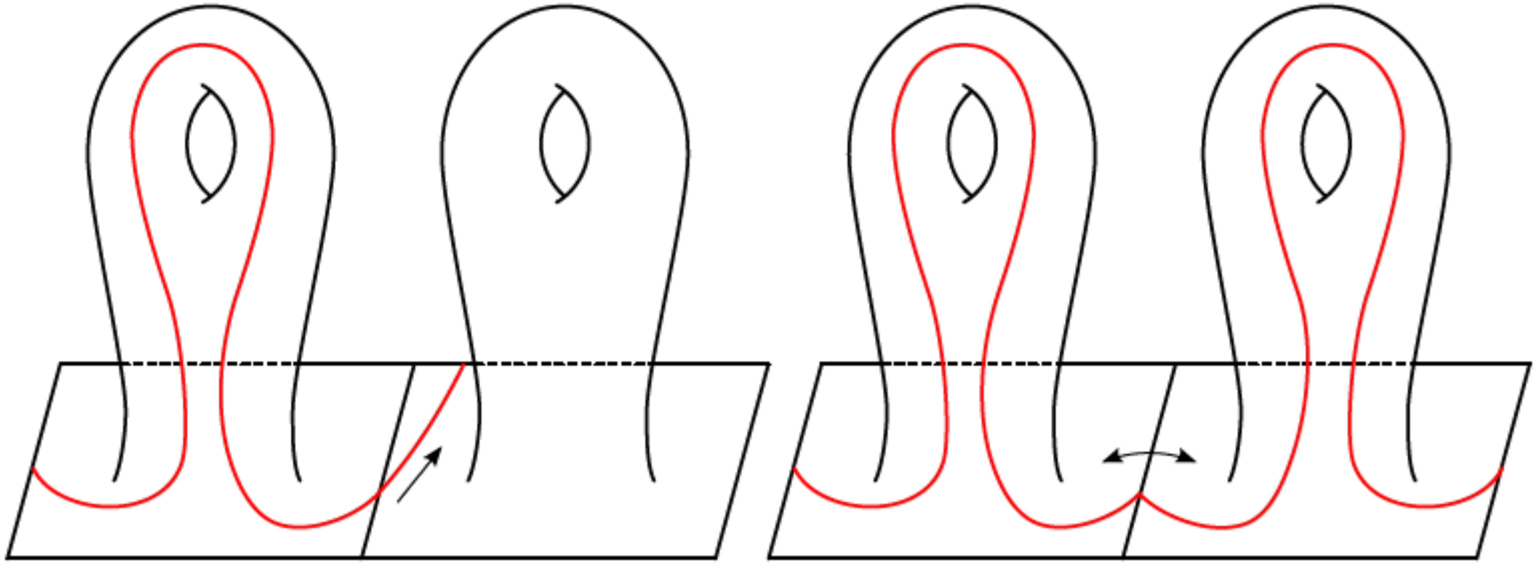
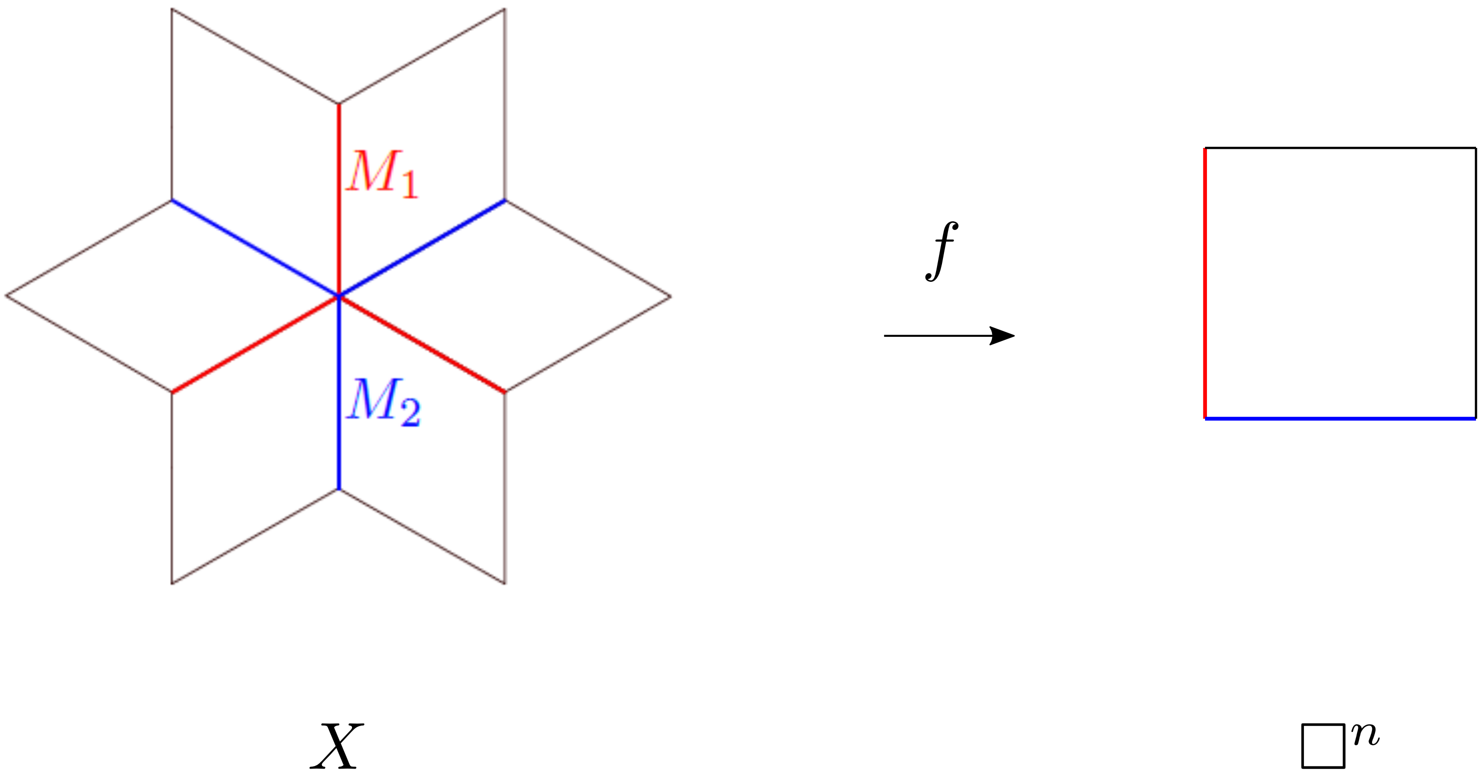


The solution: foldability and mirrors



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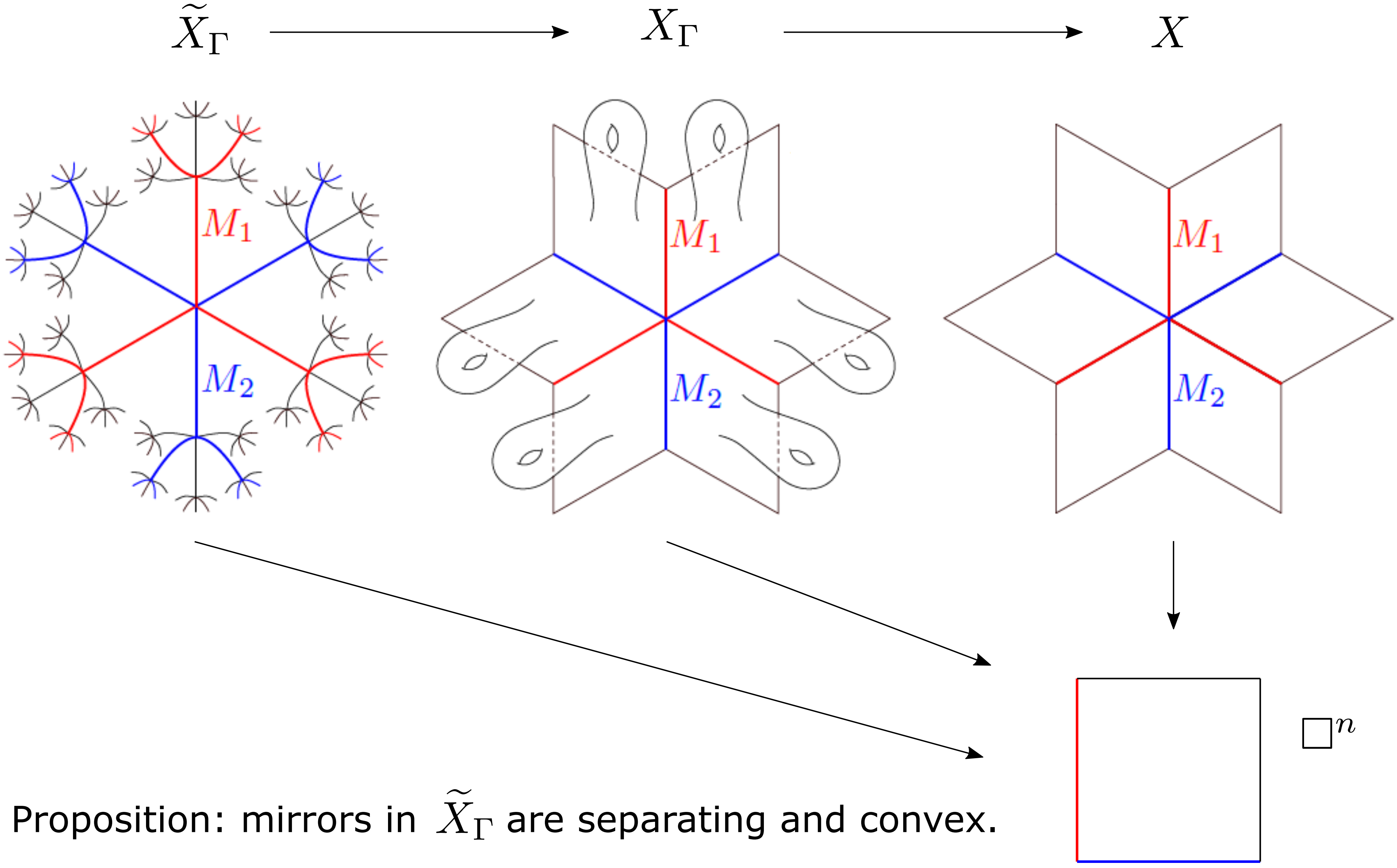
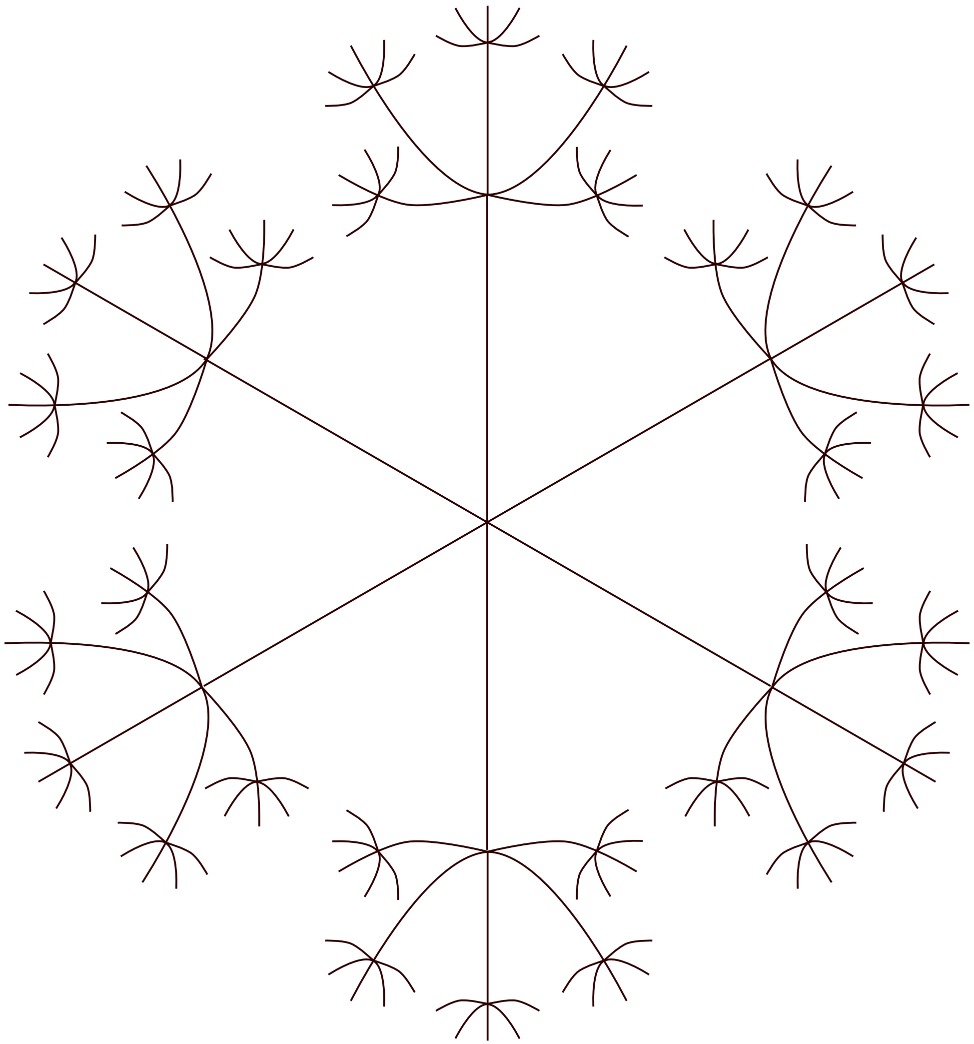
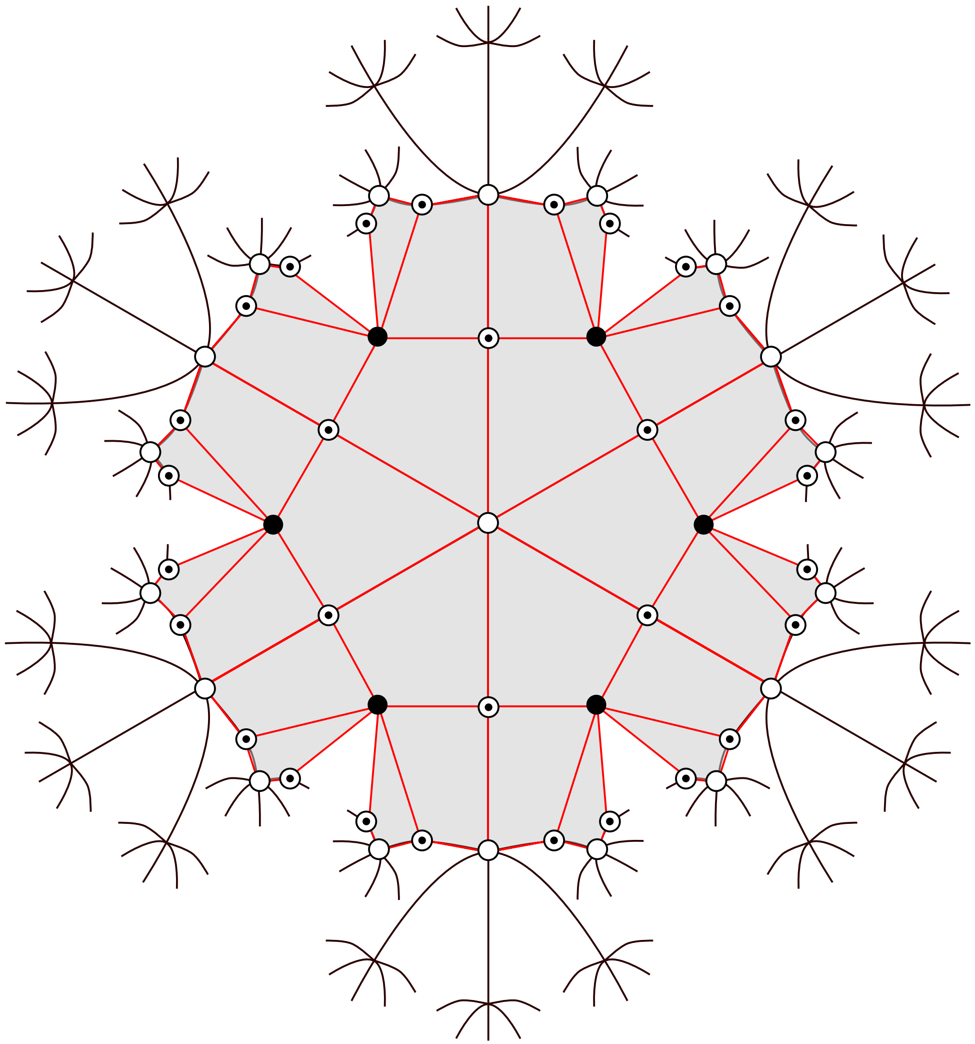


Proposition: mirrors in $\widetilde{X}_\Gamma$ are separating and convex.

The dual cube complex



$\widetilde{X}_\Gamma$



$\mathcal{C}(\widetilde{X}_\Gamma)$

The dual cube complex

vertices = k-cells

edges = codimension-1 inclusions

higher-dim cubes when needed

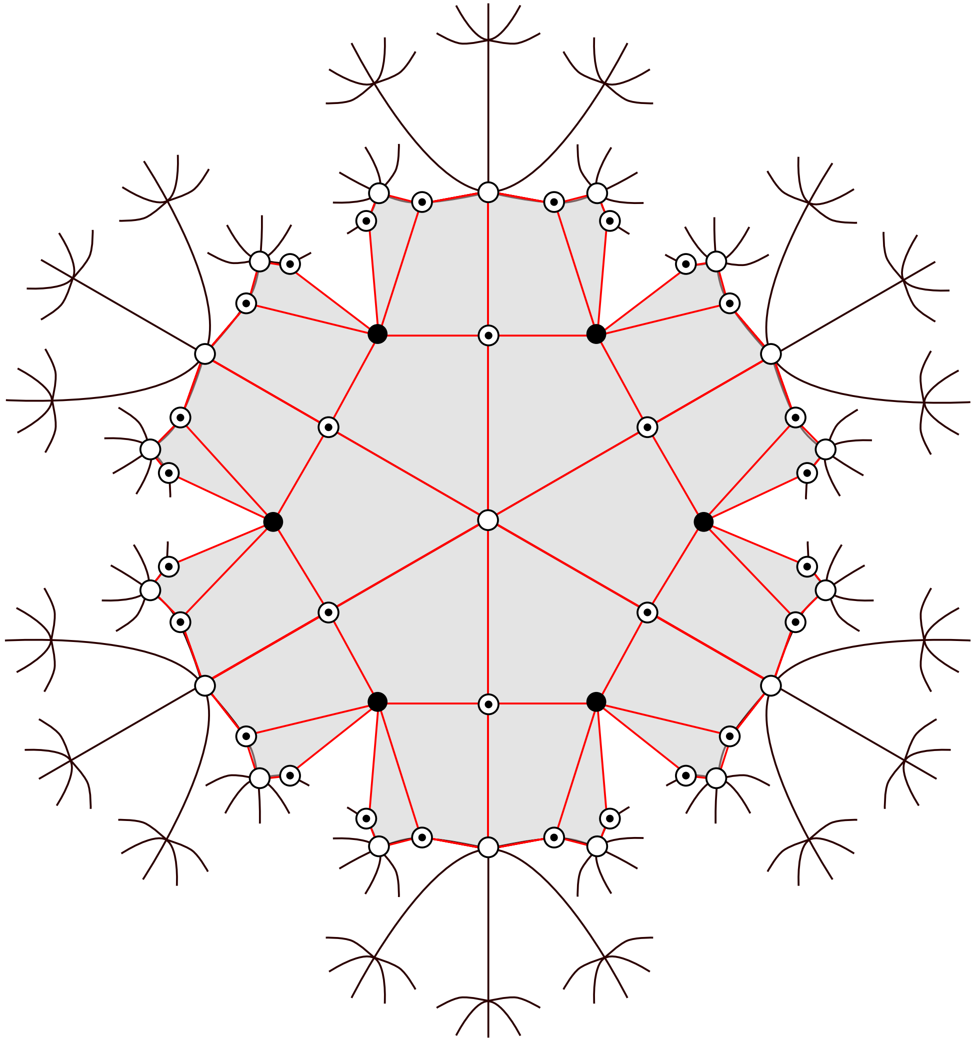
<u>Theorem</u> (Lafont, R., 2022)

Let X be a cubical complex that is

- non-positively curved,
- compact,
- foldable,
- homogeneous,
- without boundary.

Then:

- $\mathcal{C}(\widetilde{X_\Gamma})$ is a CAT(0) cubical complex.
- $\pi_1(X_\Gamma) \curvearrowright \mathcal{C}(\widetilde{X_\Gamma})$ cocompactly by isom.



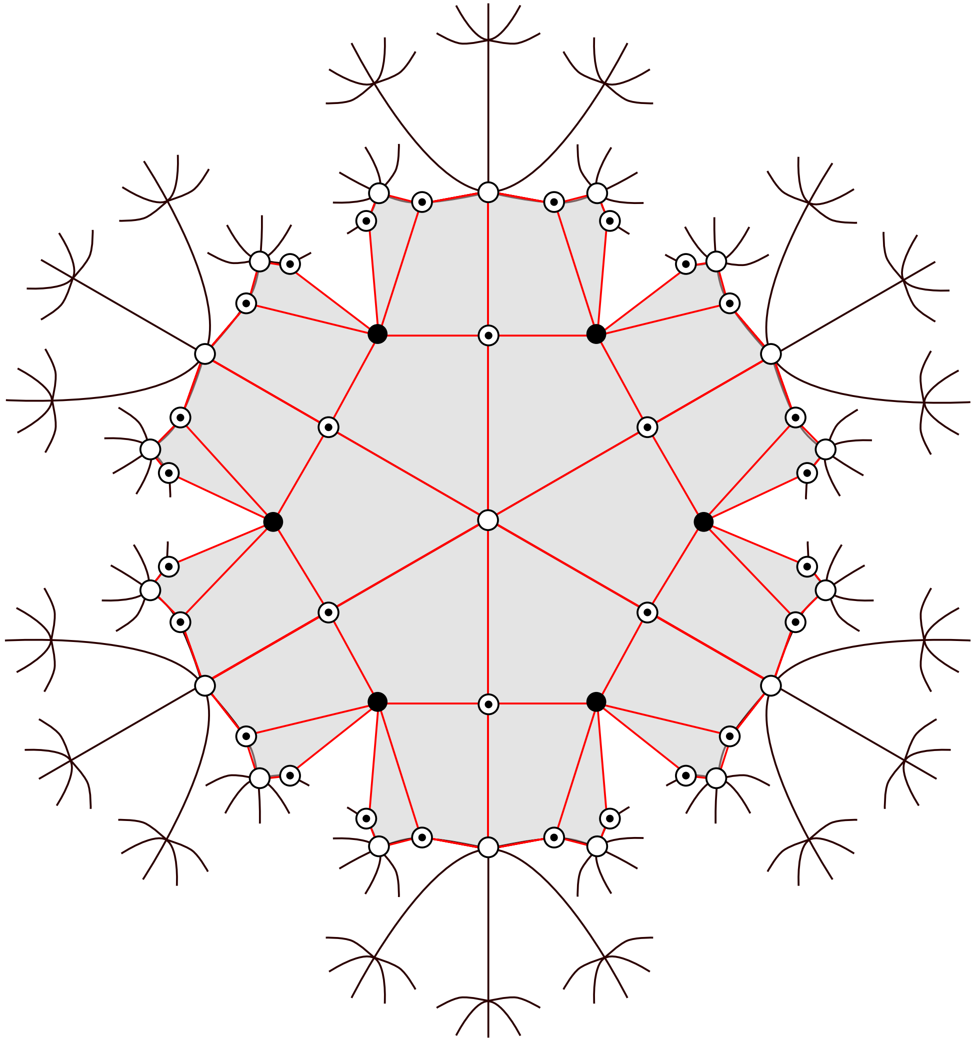
$\mathcal{C}(\tilde{X}_\Gamma)$

The dual cube complex

$\pi_1(X_\Gamma) \curvearrowright \mathcal{C}(\widetilde{X_\Gamma})$

- not locally finite
- not proper
- nice cube stabilizers (Groves-Manning 2018)

<u>Theorem</u> (Lafont, R., 2022)
Let X be as before. Then $\pi_1(X_\Gamma)$ is hyperbolic and virtually compact special.



$\mathcal{C}(\tilde{X}_\Gamma)$

Classical applications of strict hyperbolization

The following can now be obtained so that the group is virtually compact special, hence in a **RAAG**, **linear** over the integers, **residually finite**, ...

- Every triangulable manifold is cobordant to a negatively curved one (Charney, Davis, 1995).

- There exist negatively curved manifolds/groups with ideal boundary not homeomorphic to S^n (Davis, Januskiewicz, 1991 + C.-D. 1995).

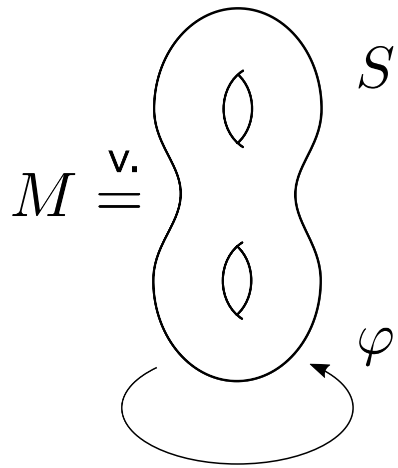
- There exist negatively curved Riemannian manifolds not homotopy equivalent to a locally symmetric manifold of rank 1 (Ontaneda 2020).

- For any triangulable manifold M, there is a hyperbolic group with ideal boundary the tree of manifolds defined by M (Swiatkowski 2020).

New applications of strict hyperbolization

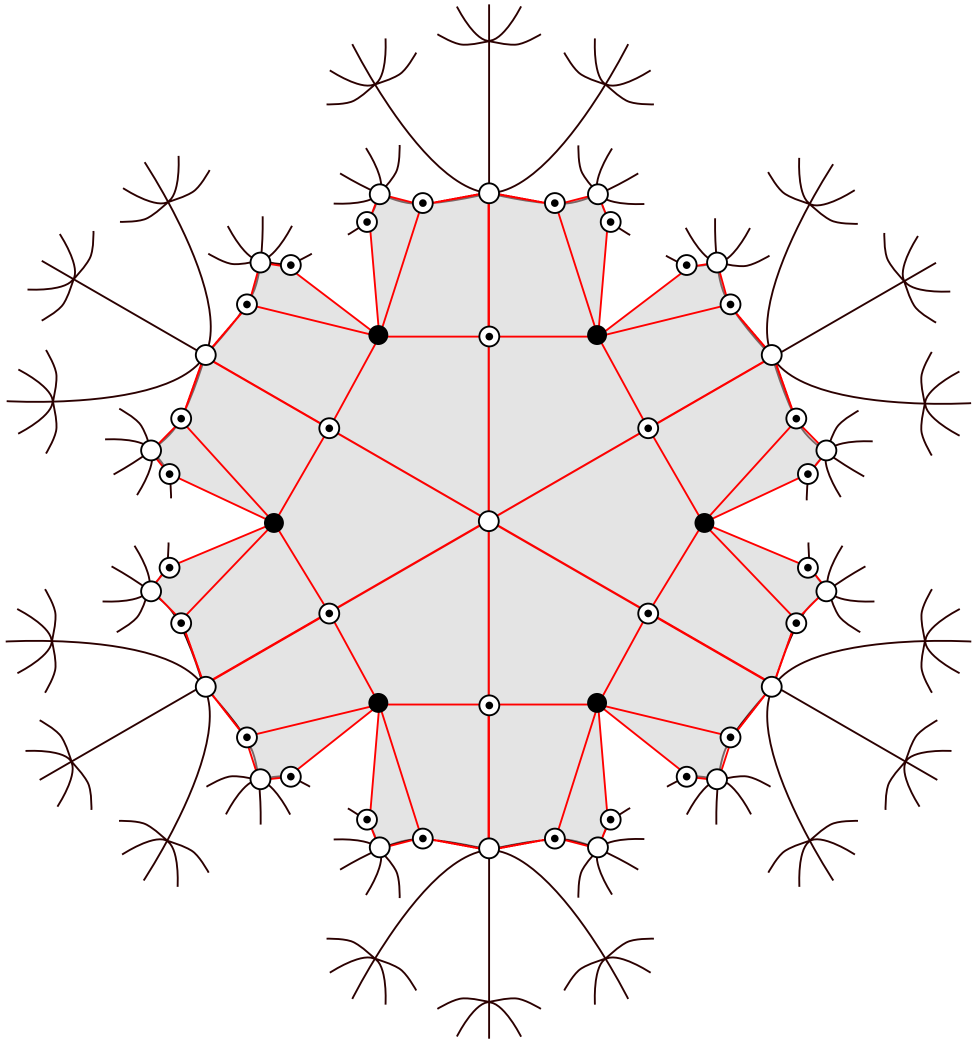
Corollary (Lafont, R., 2022)
Construction of new hyperbolic groups that virtually algebraically fiber.

$$1 \to K \to G \to \mathbb{Z} \to 1$$



Ex: M closed hyperbolic 3-manifold
(Stallings, Thurston, Agol, ...)

Proof: let X be a suitable cubulation of a smooth manifold.
Ontaneda: for suitable Γ, X_Γ carries a Riemannian metric with sectional curvatures in $[-1-\varepsilon, -1]$.
Donnelly-Xavier 1984: $b_1^{(2)}(X_\Gamma) = 0,$ for ε small enough.
Our result + Agol 2008: $\pi_1(X_\Gamma)$ is virtually RFRS.
Kielak 2020: $\pi_1(X_\Gamma)$ virtually algebraically fibers. $\square$



Thank you.