Baumslag-Solitar groups, automorphisms and generalisations

(joint with Sam Hughes)

Naomi Andrew

University of Southampton

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Baumslag–Solitar groups

These are a family of two-generator, one-relator groups with presentations:

$$\mathrm{BS}(n, m) = \langle a, t : t^{-1} a^n t = a^m \rangle$$

> *These [groups] destroy several conjectures long and strongly held by the reviewer and others, and some published and unpublished "theorems".*
>
> — BH Neumann, AMS review

Theorem (Moldavanskii '91)

$\mathrm{BS}(n, m) \cong \mathrm{BS}(p, q)$ *if and only if* $\{n, m\} = \{p, q\}$ *or* $\{-p, -q\}$

Take the convention (allowed by this theorem) that $n > 0$, and $|m| \geq n$.

Baumslag–Solitar groups as counterexamples

Theorem (Baumslag–Solitar '62, Meskin '72)

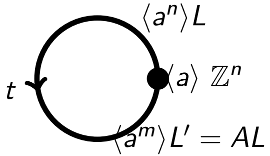
- $\mathrm{BS}(n, m)$ *is* **residually finite** *if and only if* $|n| = |m|$, *or* $n = 1$.
- $\mathrm{BS}(n, m)$ *is* **Hopfian** *if and only if* $n = 1$ *or* n *and* m *have the same prime factors.*

A group G is residually finite if for every non-identity element g, there is a map $G \to F$, where F is finite and g is not in the kernel.
A group G is Hopfian if every surjective homomorphism $G \to G$ is also injective.
So, for instance:

- $\mathrm{BS}(1, 2)$ and $\mathrm{BS}(2, -2)$ are residually finite (and Hopfian)
- $\mathrm{BS}(2, 3)$ is not Hopfian (or residually finite)
- $\mathrm{BS}(12, 18)$ is Hopfian but is not residually finite

Another way to think of Baumslag–Solitar groups is as HNN extensions of $\mathbb{Z}$.

One generalisation is to take take HNN extensions of $\mathbb{Z}^n$, not just of $\mathbb{Z}$.



They are "controlled" by a matrix over $\mathbb{Q}$: let $A \in GL_N(\mathbb{Q})$, and $L \leq \mathbb{Z}^n \cap A^{-1}\mathbb{Z}^n$ (finite index), then the presentation is:

$$\mathrm{LM}(A, L) = \langle x_1, \ldots x_n, t : [x_i, x_j], t^{-1}yt = Ay \text{ for } y \in L \rangle$$

Leary–Minasyan groups

Unsurprisingly, these "higher rank" analogues have some similar behaviour to $\mathrm{BS}(n, m)$:

Theorem (Leary–Minasyan '21)

$\mathrm{LM}(A, L)$ *is residually finite if and only if either* $L = \mathbb{Z}^n$ *or* $AL = \mathbb{Z}^n$*, or* A *is conjugate (in* $\mathrm{GL}_n(\mathbb{Q})$*) to an element of* $GL_n(\mathbb{Z})$*.*

They also have some new tricks: for instance $A = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$ and $L = \left\langle \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle$ gives an $\mathrm{LM}(A, L)$ that is $\mathrm{CAT}(0)$ but not biautomatic.

Automorphisms of groups

Let G be a group.

$\operatorname{Aut}(G) = \{\text{isomorphisms } G \to G\}$ is its *automorphism group*

$\operatorname{Inn}(G) = \{\text{those induced by conjugations}\}$ are the *inner automorphisms*. This group is isomorphic to $G/Z(G)$.

$\operatorname{Out}(G) = \operatorname{Aut}(G)/\operatorname{Inn}(G)$ are the *outer automorphisms*.

Theorem (Collins, Collins–Levin, Gilbert et al, (Clay))

The outer automorphims of a Baumslag–Solitar group satisfy:

- *if* $n \mid m$, *and* $n, |m|, |m/n| \neq 1$, *then* $\mathrm{Out}(\mathrm{BS}(n,m))$ *is not finitely generated,*
- $\mathrm{Out}(\mathrm{BS}(1,m))$ *is finitely generated, but infinite provided* m *is composite,*
- $\mathrm{Out}(\mathrm{BS}(n,n)) \cong D_\infty \times C_2$,
- $\mathrm{Out}(\mathrm{BS}(n,-n)) \cong D_{2n} \times C_2$,
- *if* n *does not divide* m, *then* $\mathrm{Out}(\mathrm{BS}(n,m)) \cong D_{|n-m|}$.

For instance: $\mathrm{Out}(\mathrm{BS}(2,4))$ is not finitely generated; but $\mathrm{Out}(\mathrm{BS}(2,3))$ has order 2.

What might we expect to happen here?

- we need to account for "small" and "huge" automorphism groups,
- it looks like there will be significant dependence on the matrix A,
- maybe different things will happen if L (or AL) is all of $\mathbb{Z}^n$.

For the rest of the talk: I will try and give an idea of the theorems we get and how to prove them.

The "small" case

Theorem (A.–Hughes)

Suppose neither L nor AL is contained in the other. Then $\mathrm{Out}(\mathrm{LM}(A, L))$ *is finitely generated, and finite according to conditions only on A.*

The proof follows similar lines to Gilbert et al, but with more Bass–Serre theoretic machinery behind it.

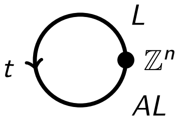
Ideas in the proof

Step 1: the deformation space is just a single point (follows from Levitt '05) – given by the graph of groups for the original presentation.

Step 2: apply Bass–Jiang's theorem on outer automorphisms fixing an action on a tree.

This is made slightly easier since all the stabilisers are abelian.

Take $A = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$, $L = \left\langle \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle$.



The outcome is that $\mathrm{Out}(\mathrm{LM}(A, L)) \cong (C_2 \times C_2) \rtimes (C_4 \rtimes C_2)$.

The generators (in reverse order) are

$$\Phi : \begin{cases} a \mapsto b \\ b \mapsto a \\ t \mapsto t^{-1} \end{cases} \quad \Psi : \begin{cases} a \mapsto b^{-1} \\ b \mapsto a \\ t \mapsto t \end{cases} \quad \Delta_1 : \begin{cases} a \mapsto a \\ b \mapsto b \\ t \mapsto ta \end{cases} \quad \Delta_2 : \begin{cases} a \mapsto a \\ b \mapsto b \\ t \mapsto tb \end{cases}$$

The “huge” case

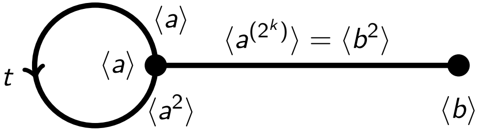
Theorem (A.–Hughes)

Suppose AL is contained in L as a subgroup with prime index, and L is not all of $\mathbb{Z}^n$. Then $\mathrm{Out}(LM(A, L))$ is not finitely generated.

The idea is to generalise Clay’s proof for Baumslag–Solitar groups.

The deformation space is much much much worse...

There are more graphs of groups, looking like (for $BS(2, 4)$):



Ideas in the proof

However, it does equivariantly deformation retract to a tree, so Bass–Serre theory is back on the table.

One still has to calculate the stabilisers (again, this is $\mathrm{BS}(2,4)$):

$$C_2 \times C_2 \underset{C_2}{\overline{\qquad\qquad}} \varinjlim C_{2^k} \rtimes C_2$$

Thanks!

Any questions?