Canonical Forms for Free Group Automorphisms

$\mathrm{Out}(\mathbb{Z}^n) \cong GL_n\mathbb{Z} \rightsquigarrow$ Jordan Canonical Form

$\mathrm{Out}(\pi_1(S)) \cong MCG^{\pm}(S) \rightsquigarrow$ Nielsen-Thurston Decomposition

$\mathrm{Out}(F) \rightsquigarrow$???

I) The Topmost Limit Tree

"singular length measures"

Thm (w.i.p.)

F f.g. free group

$\phi : F \to F$ automorphism

$\Rightarrow$ $\exists$ $\mathbb{R}$-tree $(T, d_1 \oplus \cdots \oplus d_k)$ s.t.

① $F \overset{\text{isom.}}{\curvearrowright} (T, \bigoplus_{i=1}^{k} d_i)$ is minimal and has trivial arc stabilizers;

$h(x \cdot p) = \phi(x) \cdot h(p)$ $\forall x \in F, p \in T$

② $\exists$ ϕ-equivariant dilation h on $(T, \bigoplus_{i=1}^{k} d_i)$; and

③ $x \in F$ is T-loxodromic $\Leftrightarrow$ $[\phi^n(x)]_{n=1}^{\infty}$ limits to a topmost attracting lamination. ⊛

dilation: $\exists$ $\lambda_1, \ldots, \lambda_k > 1$ s.t. h is a λ_i-homothety wrt d_i

Moreover, $(T, \bigoplus_{i=1}^{k} d_i)$ is "unique". $\leftarrow$ $(T, \bigoplus_{i=1}^{k} s_i d_i)$, $s_i > 0$

Rmk: T is a point $\Leftrightarrow$ ϕ is polynomially growing : $\forall x \in F$, $\|\phi^n(x)\|$ grows polynomially as $n \to \infty$.

Otherwise, ϕ is exponentially growing.

II) Known Cases

A) Geometric : $\phi : F \to F$ is induced by a surface homeo.

↳ Nielsen-Thurston Decomposition '70s

B) Irreducible : ϕ has no invariant proper free factor system

↳ Bestvina-Feighn-Handel '96

Other Canonical Forms

1) Linearly growing [Cohen-Lustig '95]

2) Quadratically growing [Lustig-Ye '17]

III) Previous Result

Thm* (preprint)

F f.g. free group

$\phi : F \to F$ automorphism

$\Rightarrow$ $\exists$ real pretree Y s.t.

① $F \overset{\text{rigid}}{\curvearrowright} Y$ is minimal and has trivial arc stabilizers;

② $\exists$ ϕ-equivariant "F-expanding" homeo. $f : Y \to Y$

③ $x \in F$ is T-loxodromic $\Leftrightarrow$ $\|\phi^n(x)\|$ grows exponentially as $n \to \infty$

Rmk: Y is a point $\Leftrightarrow$ ϕ is polynomially growing

$F \curvearrowright Y$ is free $\Leftrightarrow$ ϕ is atoroidal

IV) Example: Let $F = F(a,b,c,d,e,f)$

Define $\phi : F \to F$ by

$$\begin{aligned} a &\mapsto ab \\ b &\mapsto bab \\ c &\mapsto cd \\ d &\mapsto dcd[a,b] \\ e &\mapsto efc \\ f &\mapsto fef \end{aligned}$$

$$a \mapsto ab \mapsto abbab \mapsto abbabbababbab \mapsto \cdots : abbabbababbabbab\ldots \sim \Lambda_3$$

$$d \mapsto dcd[a,b] \mapsto dcd[a,b]cddcd[a,b][a,b] \mapsto \cdots : dcd[a,b]cddcd[a,b]^2cddcd[a,b]d\ldots \sim \Lambda_2$$

$$e \mapsto efc \mapsto efcfefcd \mapsto efcfefcdfefefcfefcddcd[a,b] \mapsto \cdots \sim \Lambda_1$$

Poset of "attracting laminations":

$$\begin{array}{ccc} \Lambda_1 & \quad & \Lambda_3 \\ \mid & & \\ \Lambda_2 & & \end{array}$$

$\leftarrow$ topmost

[Bestvina-Feighn-Handel '00]

V) Proof by Example: F & ϕ as before.

Set $F_1 := F$, $F_2 := \langle a, b, c, d \rangle \leq F_1$, and $F_3 := \langle a, b \rangle \leq F_2$.

Each F_i is ϕ-invariant and $\phi_i := \phi|_{F_i}$ is irreducible "rel. F_{i+1}"

"Bestvina-Feighn-Handel" $\rightsquigarrow$ $\exists$ $\mathbb{R}$-tree (T_i, d_i) s.t.

① $F_i \overset{\text{isom.}}{\curvearrowright} (T_i, d_i)$ is minimal and has trivial arc stab.

② $\exists\, \lambda_i > 1$ and ϕ_i-equivariant λ_i-homothety h_i on (T_i, d_i)

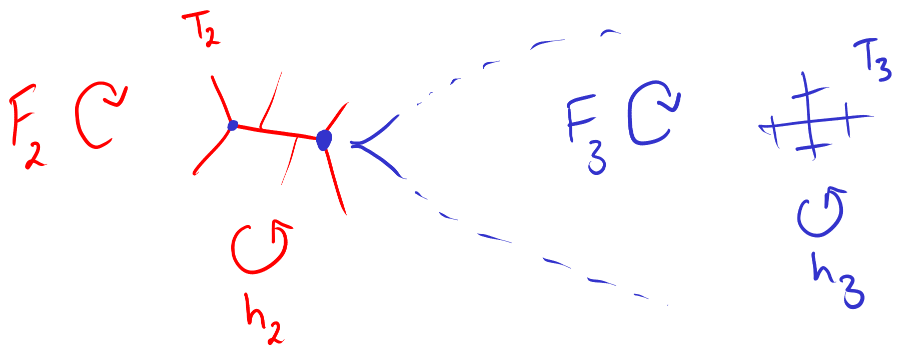
③ $x \in F_i$ is T_i-lox. $\Leftrightarrow$ $[x]$ has exp. growth rel. F_{i+1}
$\Leftrightarrow$ $[\phi^n(x)] \xrightarrow[n\to\infty]{}$ att. lam. Λ_i

$$\begin{cases} T_1\text{-pt. stab.} = [F_2] \text{ in } F_1 \\ T_2\text{-pt. stab.} = [F_3] \text{ in } F_2 \\ T_3\text{-pt. stab.} = [\langle [a,b] \rangle] \text{ in } F_3 \end{cases}$$

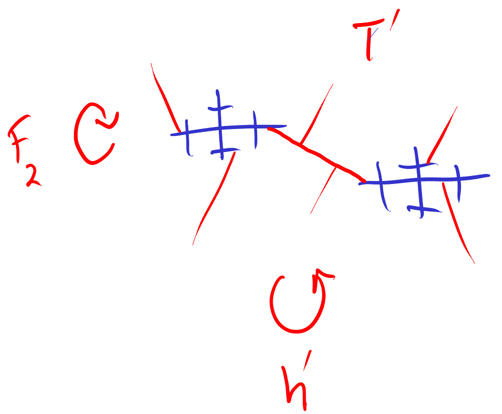
- (T_3, d_3) is <u>the</u> topmost limit tree for $\phi_3 : F_3 \to F_3$.

- In F_2, both Λ_2 & Λ_3 are the topmost att. lam.

  $\Rightarrow$ (T_2, d_2) is <u>not</u> a topmost limit tree for $\phi_2 : F_2 \to F_2$!



⊛ $\Downarrow$ blow-up

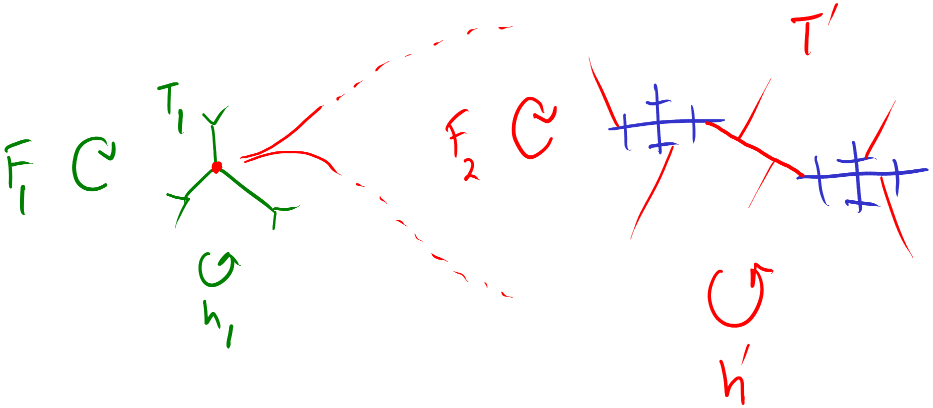


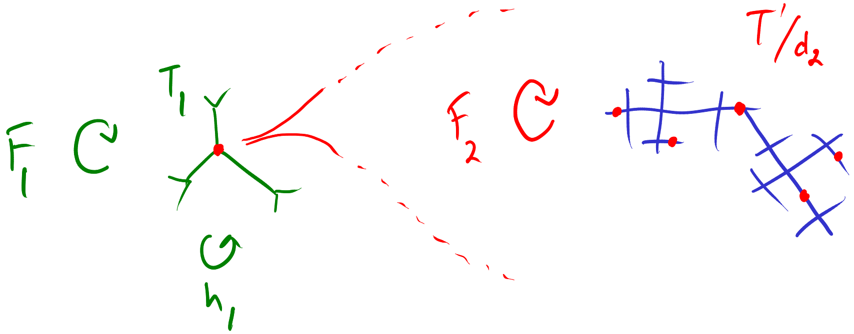
$(T', d_2 > d_3)$ + Λ_2 & Λ_3 topmost

⊛ $\Downarrow$

$(T', d_2 \oplus d_3)$ is a topmost limit tree for ϕ_2.

• In F_1, Λ_1 & Λ_3 are the topmost att. lam.





$$(T'', d_1 \succ d_2 \oplus d_3)$$

$$\Downarrow ?$$

$$(T'', d_1 \oplus d_2 \oplus d_3)$$

Fails: $\begin{array}{c}\Lambda_1 \\ | \\ \Lambda_2\end{array}$ and $\lambda_2 \leq \lambda_1$.

$(T, d_1 \succ d_3)$ + Λ_1 & Λ_3 are topmost

$$\Downarrow$$

$(T, d_1 \oplus d_3)$ is a topmost limit tree for $\phi_1 = \phi$.

□