On finitely generated normal subgroups of right-angled Artin groups

Joint work with Montserrat Casals-Ruiz (UPV/EHU).

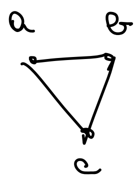
<u>Def</u>: Let X be a finite simplicial graph. The right-angled Artin group (RAAG) associated to X, denoted by GX, is given by the following presentation:

$$GX = \langle \underbrace{V(X)}_{\text{vertex set}} \mid xy = yx \iff x, y \text{ adjacent in } X \rangle$$

<u>Ex</u>:

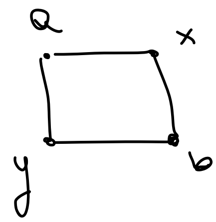
1) $X = \overset{\cdot}{a} \quad \overset{\cdot}{b} \quad \overset{\cdot}{c} \rightsquigarrow GX = \langle a, b, c \mid \emptyset \rangle = F_3$

The RAAG associated to a totally disconnected graph with n vertices is F_n.

2) $X =$ 

$\rightsquigarrow GX = \langle a, b, c \mid ab = ba, bc = cb, ac = ca \rangle = \mathbb{Z}^3$

The RAAG associated to a complete graph with n vertices is $\mathbb{Z}^n$.

3) $X = C_4 =$ 

$\rightsquigarrow$

$$\rightsquigarrow \quad GC_4 = \langle a, b \mid \emptyset \rangle \times \langle x, y \mid \emptyset \rangle = F_2 \times F_2$$

The direct product of finitely many RAAGs is a RAAG.

Also: The free product of finitely many RAAGs is a RAAG.

MOTIVATION (Why do we care about finitely generated normal subgroups of RAAGs?)

$\boxed{1}$ F.g. normal subgroups of free groups

Thm (Schreier, 1928): If F is a free group and N is a non-trivial finitely generated normal subgroup of F, then N is of finite index in F.

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Also true if

- F is a free product (B. Baumslag, 1966).

- F is a finitely presented group with $\operatorname{def}(F) > 1$. (Bieri, Neumann, Strebel, 1987).

Def (deficiency): Let $\mathcal{P} = \langle X \mid R \rangle$ be a finite presentation of a finitely presented group G. The difference $|X| - |R|$ is the deficiency of $\mathcal{P}$, $\operatorname{def}(\mathcal{P})$ and

$$\operatorname{def}(G) = \sup \{ \operatorname{def}(\mathcal{P}) \mid \mathcal{P} \text{ presentation of } G \}$$

is called the deficieny of G.

- F is finitely presented with $\beta_1^{(2)}(F) \geq 1$ (Gaboriau, 2002)

$\boxed{2}$ Algorithmic problems.

The algorithmic problems are decidable in the class of f.g. subgroups of free groups

But when we take direct products (still in the class of RAAGs) ...

<u>Thm</u> (Mihailova, 1958): $\exists H_1 < F_2 \times F_2$ f.g. with undecidable membership problem in $F_2 \times F_2$

<u>Thm</u> (Miller, 1971): $\exists H_2 < F_2 \times F_2$ f.g. with undecidable conjugacy problem.

<u>Thm</u> (Grunewald, 1978): H_1, H_2 are not f.p.

Thm (Baumslag - Roseblade, 1984): If $\underset{\text{f.p.}}{S} < F \times F'$, then

- S is free; or
- S is virtually the direct product of two free groups.

Cor (Baumslag - Roseblade, 1984): The conjugacy and the membership problems are decidable in the class of f.p. subgroups of the direct product of two free groups.

Thm (Bridson, 2012): $\exists\ \underset{\text{f.p.}}{H} < A \times A$ such that the conjugacy and the membership problems are undecidable in H.

A RAAG

Q: For which f.g. subgroups of RAAGs we have that the membership and the conjugacy problem are decidable?

$\boxed{3}$ Work of Bridson, Howie, Miller, Short on subgroups of direct products of limit groups (e.g., free groups, surface groups).

1) Generalise the work of Baumslag-Roseblade to f.p. subgroups of direct products of free groups.

2) F.g. / F.p. residually free groups, then

$$S < \underbrace{\Gamma_1 \times \dots \times \Gamma_n}_{\text{limit groups}}$$

f.g.

f.p.

Thm (BHMS, 2009): Let $\Gamma_1, \dots, \Gamma_m$ be non-abelian limit groups (e.g., free groups, surface groups)

Let $D = \Gamma_1 \times \dots \times \Gamma_m$ and let $S < D$ be a finitely presented subgroup (full, subdirect)

f.p. residually free group

Then,

(1) $p_{i,j}(S) \underset{f.i.}{<} \Gamma_i \times \Gamma_j$, where

$p_{i,j} : D \longrightarrow \Gamma_i \times \Gamma_j$

Virtually surjects onto pairs

(2) There is $D_0 \underset{f.i.}{<} D$ such that

$\gamma_{m-1}(D_0) < S$

If the first such term is $\gamma_{c+1}(D_0)$, then the co-nilpotency class of S is defined to be c.

Ex:

- S has co-nilpotency class 0 if it is of finite index in D.

- S has co-nilpotency class 1 if it is virtually the kernel of a map from a product of non-abelian limit groups to an infinite abelian

group (= Stallings - Bieri type).

Challenges :

1) Fully characterise f.p. subgroups of direct products of limit groups

2) Construct f.p. subgroups of direct products of free groups whose co-nilpotency class is greater than 2.

Answers :

<u>The Virtual Surjection onto Pairs Criterion</u> (BHMS, 2012): Let $G_1, \ldots, G_n$ be f.p. group and $S < G_1 \times \ldots \times G_n$ a f.g. subgroup. If virtually surjects onto pairs, then S is f.p.

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$\Downarrow$

<u>The Virtual Surjection Conjecture</u>: Let $G_1, \dots, G_n$ be groups of type F_k and and $S < G_1 \times \dots \times G_n$. Assume that S virtually surjects onto k-tuples of factors, i.e. for any $i_1, \dots, i_k$ with $1 \le i_1 < \dots < i_k \le n$ the image of S under the projection

$$G_1 \times \dots \times G_n \longrightarrow G_{i_1} \times \dots \times G_{i_k}$$

is of finite index. Then S is of type F_k.

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<u>Thm</u> (BHMS, 2012): There are f.p. subgroups of direct products of free groups with co-nilpotency ≥ 2.

MAIN RESULT

Thm

Let G_X be a RAAG and let

$G_X = G_{X_1} \times \ldots \times G_{X_n}$ be the direct product decomposition of G_X such that G_{X_i} is directly indecomposable, $i \in \{1, \ldots, n\}$.

Let N be a full non-trivial f.g. normal subgroup of G_X. Then,

$N \cap G_{X_i} \neq 1$
for all $i \in \{1, \ldots, n\}$

- G_X / N is virtually $\mathbb{Z}^n$ for some $n \in \mathbb{N} \cup \{0\}$ (abelian-by-finite)

- G_X / N is finite-by-abelian (i.e., $\exists \chi \in \Sigma^1(G_X)$ s.t. $N \underset{f.i.}{<} \operatorname{Ker} \chi$)

The theorem is true for a more general setting:

(1) It holds also for RACGs.

(2) It holds for a graph product $\mathcal{G}(X, \{G_i\}_{i \in V(X)})$ such that the graph X does not have any universal vertex (i.e. a vertex that is adjacent to all the other vertices of the graph X).

(3) It holds for graph towers over RAAGs.

CONSEQUENCES

$\boxed{1}$ Algorithmic problems and conjugacy separability

<u>Def</u>: A group G is conjugacy separable if for any two non-conjugate elements $x, y \in G$ there is a homomorphism

$$\varphi : G \to \underset{\text{finite group}}{Q}$$

such that $\varphi(x)$ is not conjugate to $\varphi(y)$ in Q.

■ Conjugacy separability $\Rightarrow$ residual finiteness

Ex:

- Virtually free groups (Dyer)

- Virtually surface groups (Martino)

- Virtually polycyclic groups (Remeslennikov)

- F.p. residually free groups (Chagas - - Zalesskii)

Def: A group G is hereditarily conjugacy separable if every finite index subgroup of G is conjugacy separable.

Ex:

- Virtually free groups (Dyer)
- Virtually surface groups (Martino)
- Virtually polycyclic groups (Remeslennikov)
- F.p. residually free groups (Chagas - Zalesskii)

Fact: There is a f.p. subgroup of a RAAG which is conjugacy separable and not hereditarily conjugacy separable (Martino - Minasyan)

Thm (Minasyan, 2012): RAAGs are hereditarily conjugacy separable.

Q: Which subgroups of RAAGs are hereditarily conjugacy separable?

<u>Thm</u> (Minasyan, 2014): Let N be a f. g. normal subgroup of a RAAG G such that the quotient G/N is virtually polycyclic. Then,

(1) N is hereditarily conjugacy separable.

(2) N has decidable conjugacy problem.

<u>Cor:</u> F. g. normal subgroups of RAAGs are hereditarily conjugacy separable and have decidable conjugacy problem

Cor: Let N be a f.g. normal subgroup of a RAAG G. Then the membership problem is decidable for N in G.

$\boxed{2}$ VSP Criterion

Thm (BHMS, 2012): There are f.p. subgroups of direct products of free groups with co-nilpotency ≥ 2.

Cor: F.g. normal subgroups of direct products of free groups are of Stallings-Bieri type. $\Rightarrow$ The examples of BHMS of f.p. subgroups with co-nilpotency ≥ 2 are not normal

The Virtual Surjection Conjecture: Let

$G_1, \dots, G_n$ be groups of type F_k and and $S < G_1 \times \dots \times G_n$. Assume that S virtually surjects onto k-tuples of factors, i.e. for any $i_1, \dots, i_k$ with $1 \le i_1 < \dots < i_k \le n$ the image of S under the projection

$$G_1 \times \dots \times G_n \longrightarrow G_{i_1} \times \dots \times G_{i_k}$$

is of finite index. Then S is of type F_k.

Thm (Kochloukova 2010 + Kuckuck 2013) The Virtual Surjection Conjecture holds for subgroups of Stallings - Bieri type

Cor: The Virtual Surjection Conjecture holds for f.g. normal subgroups.

OBSERVATION

It is a very particular property of RAAGs.

For example, if we pass to f.p. subgroups of RAAGs it does not hold

Rips Construction + Haglund-Wise:

For Q a f.p. group, there is a ses

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1$$

K: f.g. normal

Γ: hyperbolic + virtually a f.p. subgroup of a RAAG.

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<u>Thm</u> (Kielak, 2020): Let G be an infinite f.g. group which is virtually RFRS. Then G is virtually fibred (it admits a f.i. subgroup mapping onto $\mathbb{Z}$ with f.g. kernel) $\iff$ $\beta_1^{(2)}(G) = 0$

Q: Another examples of classes of groups where the only way of having f.g. normal subgroups is fibring?