Irreducible lattices fibring over the circle

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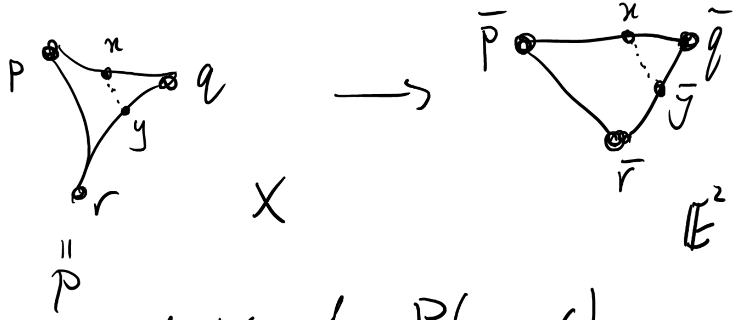
A geodesic metric space X is CAT(0) if



$\forall$ triangles $P(p,q,r)$

$\forall\ x,y \in \partial P$

we have $d_X(x,y) \leq d_{\mathbb{E}^2}(\bar{x},\bar{y})$

examples: trees, $\mathbb{E}^n$, symmetric spaces of noncompact type

most buildings

Let $H = \mathrm{Isom}(X)$, let $\Gamma < H$ be discrete then Γ is a uniform lattice if X/Γ is compact.

examples: $F_n \curvearrowright T_{2n}$

$\mathbb{Z}^n \curvearrowright \mathbb{E}^n$

$SL_2(\mathbb{Z}) \curvearrowright \mathbb{R}H^2$

Suppose $X = X_1 \times X_2$, each X_i is irreducible

$\hookrightarrow$ X_i does not decompose as $X_{i,1} \times X_{i,2}$ with each $X_{i,j}$ unbounded

$G_i = \mathrm{Isom}(X_i)$, $H = G_1 \times G_2$

We say $\Gamma \underset{\text{lattice}}{<} H$ is irreducible

if $\pi_{G_i}(\Gamma)$ is non-discrete

and Γ does not split as a direct product of two infinite groups

Margulis: If $\Gamma \underset{\substack{\text{irr} \\ \text{lattice}}}{<} \prod_{i=1}^{n} G_i$, $n \geq 2$

and G_i is a simple Lie group with finite centre, then $H^1(\Gamma) = 0$.

Recall a group G is type F_n

if $\exists$ a $K(G,1)$ with finite n-skeleton.

Q. Let Γ be a CAT(0) lattice in a nontrivial product. If $\exists$ $\underset{\neq 0}{\varphi} \in H^1(\Gamma)$ s.t. $\mathrm{Ker}(\varphi)$ is type F_n for $n \geq 1$, then is Γ reducible?

Thm [H. '22]: Let K be a flag complex on $[m]$. Let $L = \bigvee_{i=1}^{s} K$ and suppose A_L is irreducible as a RAAG. Then,

(i) $\exists\ \Gamma_L < \mathrm{Isom}(\mathbb{E}^s) \times \mathrm{Aut}(\tilde{S}_L)$ an irr. lattice

(ii) If L is $(n-1)$-connected, then $\exists\ \varphi : \Gamma_L \twoheadrightarrow \mathbb{Z}$ s.t. $\mathrm{Ker}(\varphi)$ is type F_n.

A_L is the raag on L

S_L is the Salvetti complex

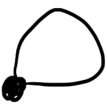
$\tilde{S}_L$ is the universal cover of S_L

Leary-Minasyan groups

Define a group LM s.t.

LM is an irr. lattice in $\mathrm{Is}(\mathbb{E}^2)\times T_{10}$

which splits as an HNN ext.



$$\left\langle \begin{bmatrix}2\\-1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix} \right\rangle^t = \left\langle \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix} \right\rangle$$

$\mathbb{Z}^2$

$$t \mapsto \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

[Leary-Minasyan '19]: LM is CAT(0)
and not biautomatic

Γ_L is in an exact sequence

$$1 \to N \to \Gamma_L \to LM \to 1$$

$L \simeq \bigvee_{i=1}^{s} K$ pick some $v \in K$
$v \neq *$

then $\exists\ A_L \overset{\pi}{\twoheadrightarrow} F_s$

$N = \mathrm{Ker}(\pi)$.

$\sim$

By calculation $\Gamma_L \curvearrowright S_L$

so $\Gamma_L \curvearrowright \tilde{S}_L \times \mathbb{E}^2$
freely cocompactly.

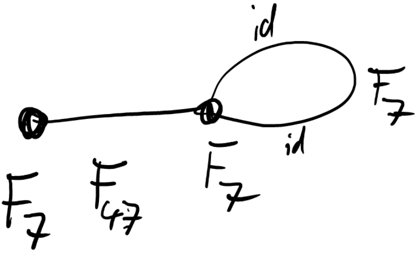
Prop: $H^1(\Gamma_L) \cong H^1(\hat{\tilde{S}}_L/\Gamma_L)$
and $\hat{\tilde{S}}_L/\Gamma_L \cong S_K$

Now do the usual Bestvina Brady argument.

Conj:

$\underset{\substack{\text{lattice} \\ \text{uniform}}}{\Gamma} \subset T_1 \times T_2$

Γ irr. $\Longleftrightarrow$ Γ does not vir. alg. fibre



$\Gamma \underset{\substack{\text{prop} \\ \text{cocomp.}}}{\circlearrowright} T_1 \times T_2 \times T_3 \simeq X$

forget $n \geq 2$ cells

then

$$\begin{tikzcd}
\pi_1(X^{(1)}) \arrow[r, hook] & \Lambda \arrow[r, two heads] \arrow[d, two heads] & \Gamma \\
& \pi_1(X^{(1)}/\Gamma)^{ab} &
\end{tikzcd}$$