Detecting Hyperbolicity in CAT(0) spaces: from Cube Complexes to Rank-Rigidity

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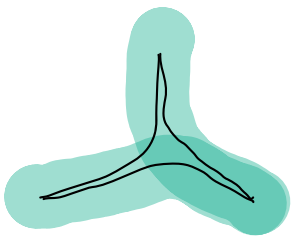
§ 1. Overview

$G \overset{\text{nice}}{\curvearrowright} X$ ← nice

$G \curvearrowright X$



- Cobounded: $\exists B$ bdd / $G.B = X$
- Metrically proper: $\forall B$ bdd, $|\{g \in G \mid gB \cap B \neq \emptyset\}| < \infty$

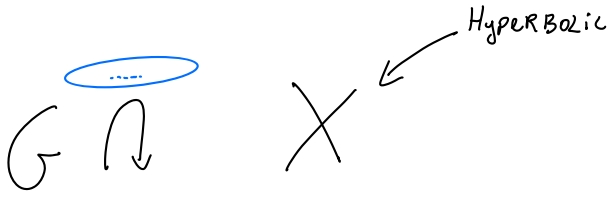
- Geodesic
- $\exists \delta$ $\forall$ geodesic triangle $\gamma_1, \gamma_2, \gamma_3$:

$$\gamma_i \subset N_\delta(\gamma_{i+1} \cup \gamma_{i-1})$$

$G \overset{\text{geometrically}}{\curvearrowright} X$ ← Hyperbolic ← negative curvature

generalize *

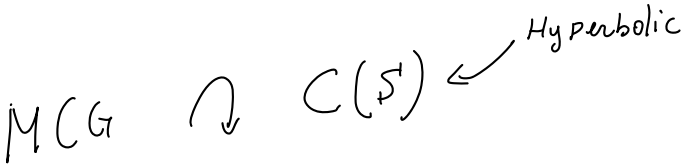
$G \overset{\text{geometrically}}{\curvearrowright} X$ ← CAT(0) ← non-positive curvature



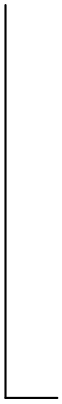
→ Relatively hyperbolic

→ Acylindrically hyperbolic

SUCCESSFULL : HAVING BOTH

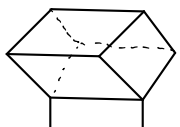


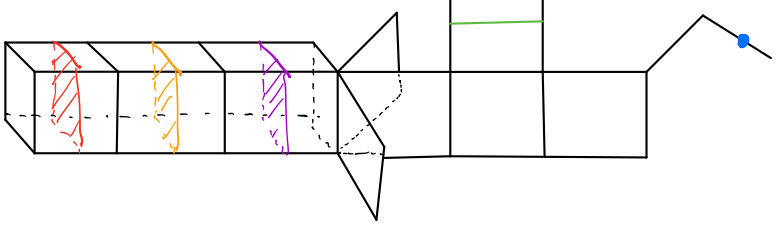
GOAL OF TODAY:

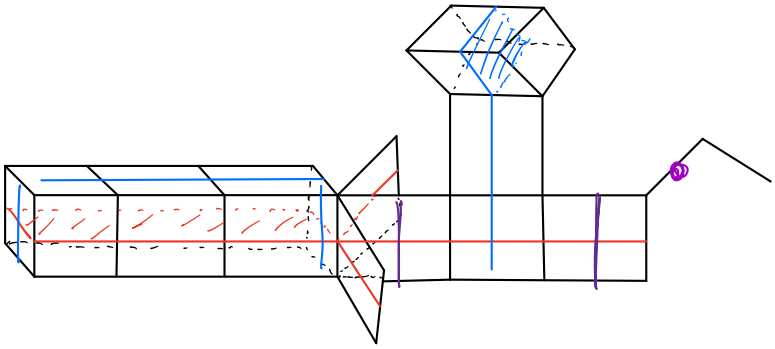


§2. WARM-UP: CAT(0) Cube Complexes

Ex 1







(Some) PROPERTIES

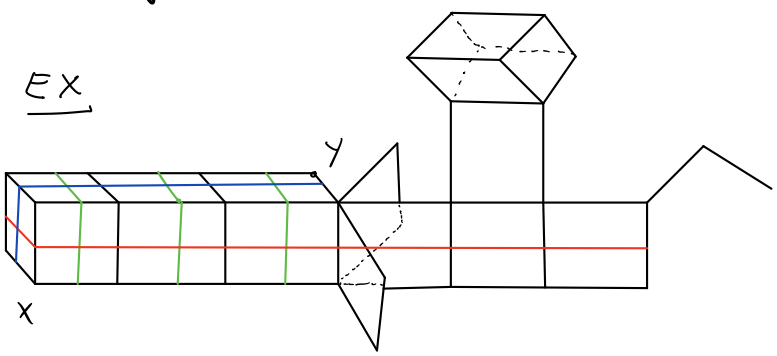
→ HYPERPLANES ARE CONVEX

→ CUT THE SPACE IN TWO COMPONENTS.

$$d_{COMBINAT}(x, y) = |\{\text{maximal collection of hyperpl separating } x \text{ from } y\}|$$

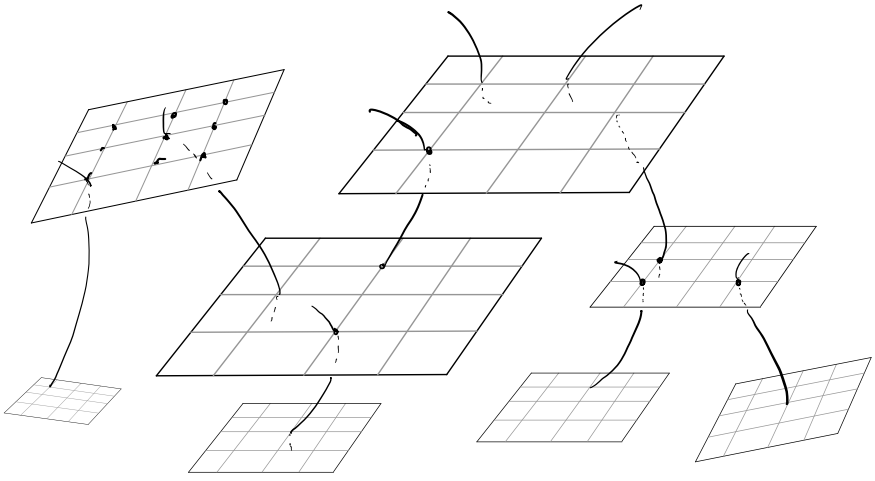
(vertices)

EX



Q: Can we use the combinatorial description to extract the "hyperbolic parts" of a C.C.C.?

Example: Tree of flats:



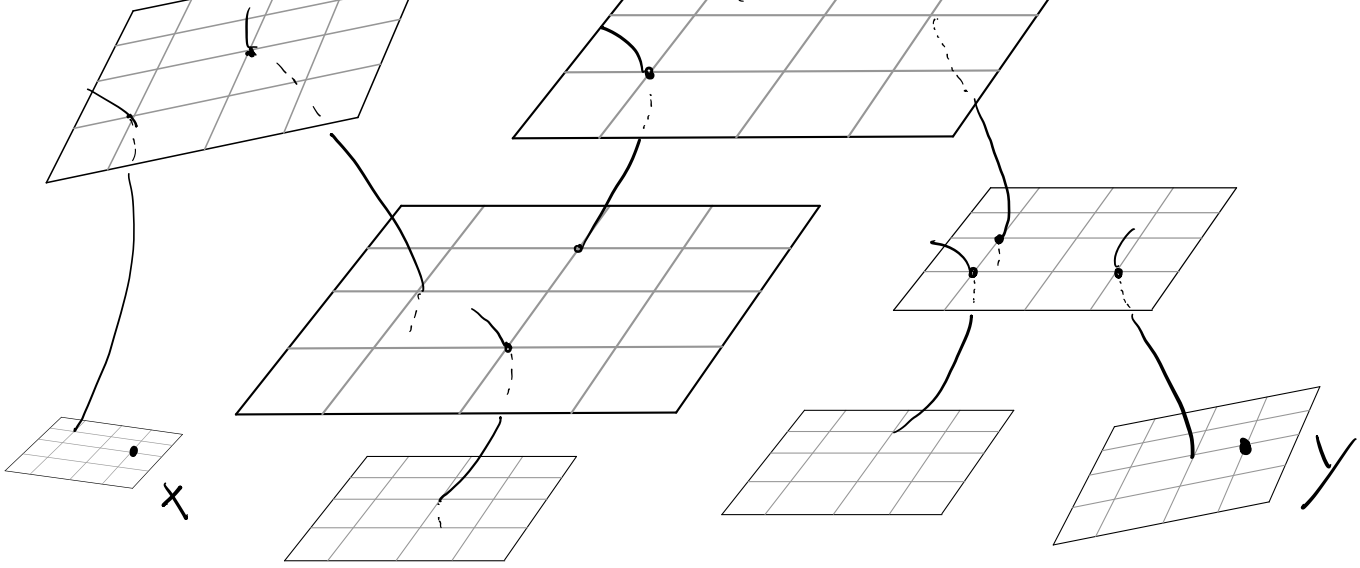
Clear separation: → Flats / → Tree part.

WANT TO COLLAPSE THE FLATS:

Definition [HAGEN] Let $(X^{(\omega)}, d_0)$ be the space w/ same vertices, but

$$d_0(x, y) = |\{\text{maximal collection of well-separated hyperplanes separating } x \text{ from } y\}|$$

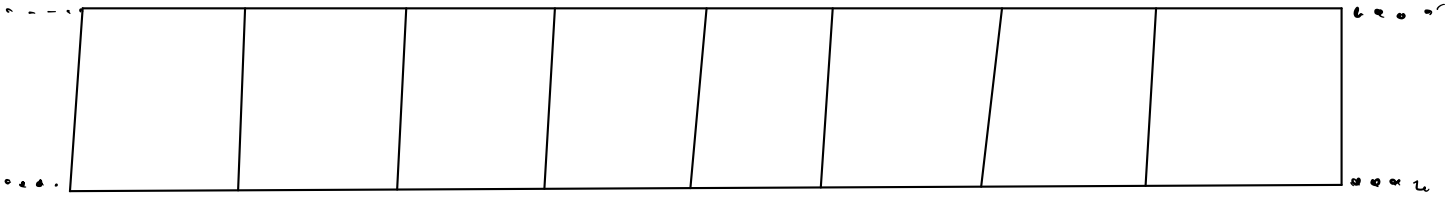
Ex



THM: X CAT(0) CUBE COMPLEX, (X, o_0) is HYPERBOLIC.

However:

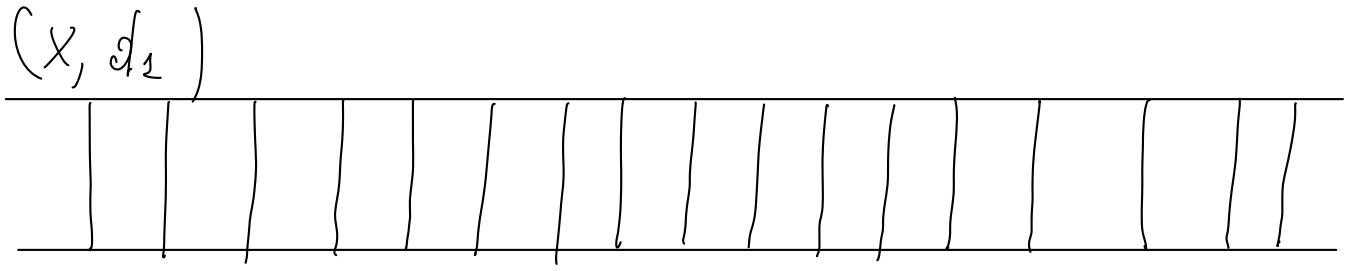
$Y =$



! (X, o_0) HAS diameter 1 (hyperbolic)

Def [Genevois]

$d_L(x,y) = |\{$maximal collection of L-well-separated hyperplanes separating x from $y\}|$

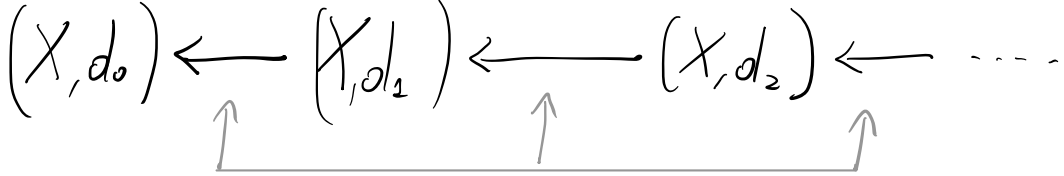


THM (X, d_L) is Hyperbolic

IDEA d_L does not collapse strips of width $\geq L$

Note L separated $\Rightarrow$ $L+1$ separated

$$d_L(x,y) \leq d_{L+1}(x,y)$$



distance-non-increasing = 1-Lipschitz.

§ 3. Back to CAT(0)

joint w/ H. Petyt, A. Zalloum (W.I.P.)

Q: How do we extract hyperbolicity without the combinatorics?

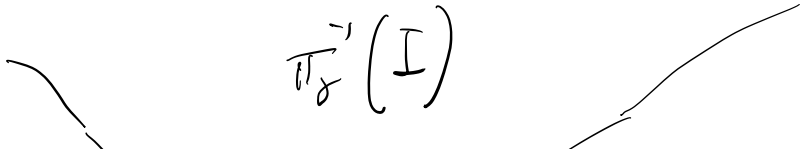
↳ How do we define "hyperplanes"?

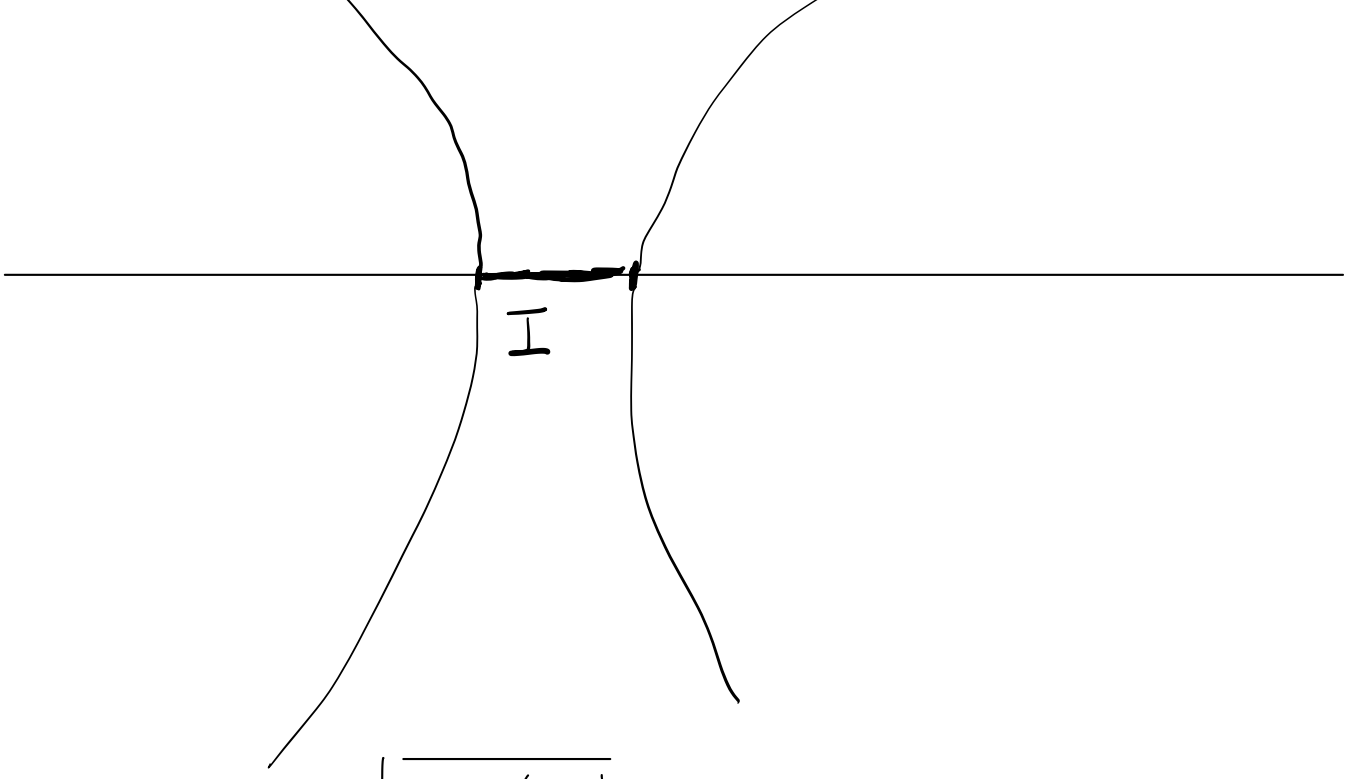
FACT X CAT(0), γ geodesic.

$\exists$ well defined $P_\gamma : X \to \gamma$ unique closest point projection

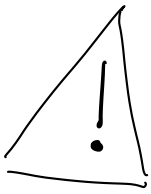
$\to$ P_γ is distance-non-increasing

Def γ a geodesic, $I \subset \mathring{\gamma}$ a unit interval,

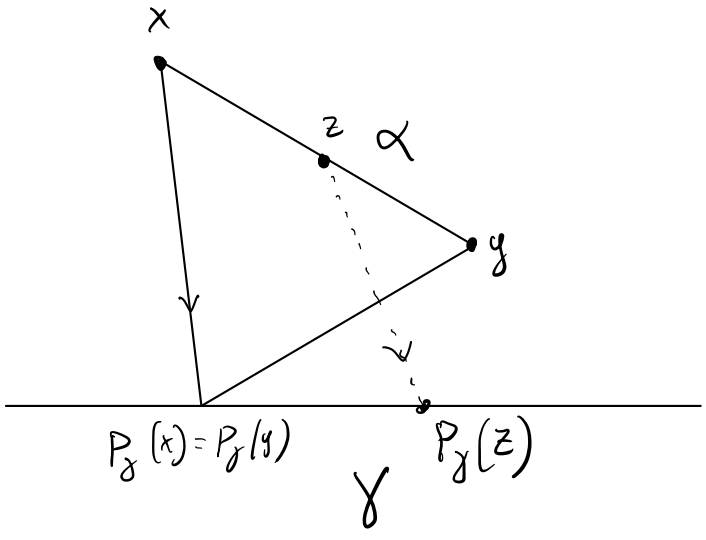


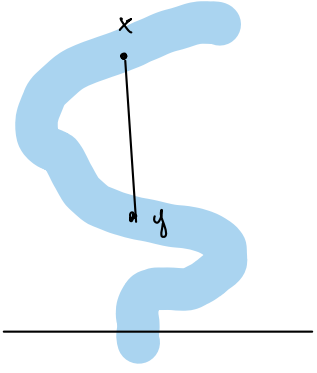


ISSUES



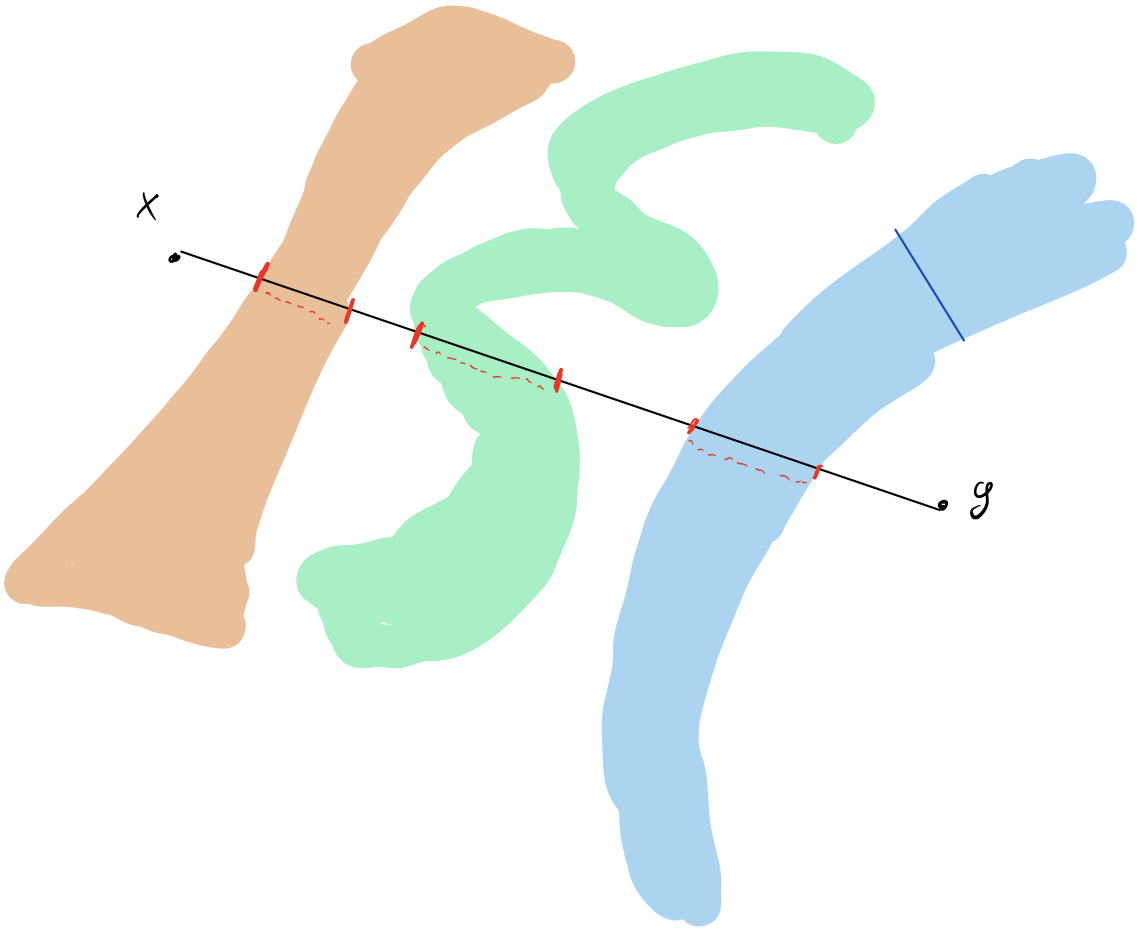
CURTAINS ARE NOT CONVEX.





THE Set of CURTAINS is UNCOUNTABLE

However: Given x, y, every <u>disjoint collection</u> of curtains separating x from y has size at most $|\mathcal{O}(x,y)|$



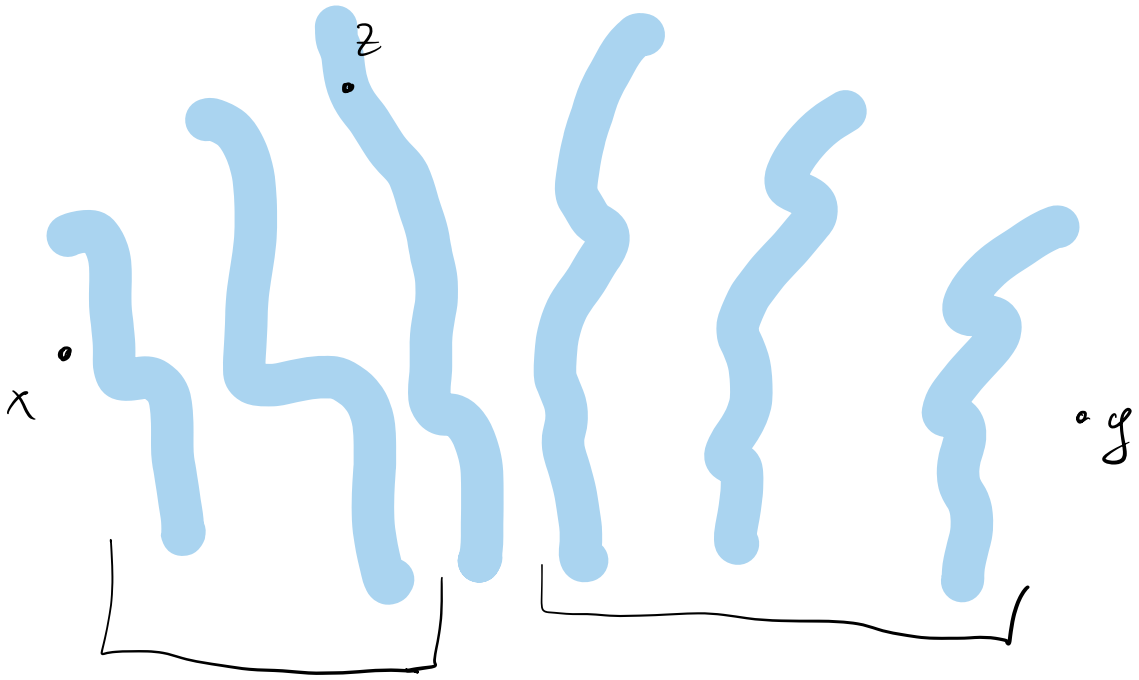
Def: X CAT(0).

$$d_L(x,y) = \left|\{\text{Largest collection of disjoint, } L\text{-SEPARATED CURTAINS SEP. } x \text{ from } y\}\right| + 1$$

(OR 0 if $x = y$.)

! This is a distance

- $0 \leq d(x,y) < \infty$
- $d(x,y) = d(y,x)$
- Triangular inequality



$d_L(x,z) = 2+1$ $\qquad$ $d_L(z,y) = 3+1$

$d_L(x,y) = 6+1$

RECALL

$$d_L(x,y) = \left| \{ \text{Largest collection of disjoint, } L\text{-SEPARATED CURTAINS SEP. } x \text{ from } y \} \right| + 1$$

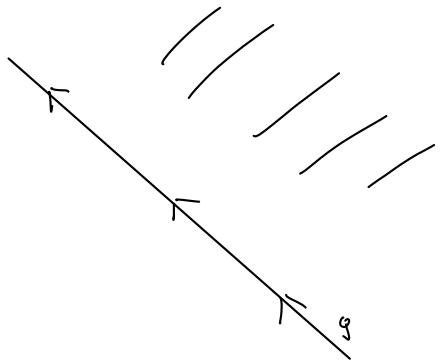
NOTE: If $G \curvearrowright X$ by isometries, $\quad G \curvearrowright (X, d_L)$ by isometries.

THM [PETYT-S-ZALLOUM]

CORO:

§4. Rank-one elements

Def: $g \in G \curvearrowright X$ CAT(0) is Rank-one if of infinite ORDER and no Axis of g bounds a half-flat



Thm [PETYT - S. ZALLOUM]

TFAE

(i) g Rk-one

(ii) There is L / $\langle g \rangle$ Q.I. embeds in (X, d_L)

(iii) An axis of g crosses an infinite, evenly distributed collection of L-sep. Contains.

COROLLARY

$G \overset{\text{geom}}{\curvearrowright} X$ CAT(0) has a RK-one element $\iff \exists L$ / (X, d_L) is UNBOUNDED

ie. $\forall$ G CAT(0) $\begin{cases} \nearrow & G \text{ has a RK-one element} \\ \searrow & (X, d_L) \text{ is Bounded } \forall L. \end{cases}$

$\rightsquigarrow$ WEAK RANK-RIGIDITY:

THM [Kent-Ricks, Petyt-S-Zalloum]

$\forall$ CAT(0) GROUP G ↗ G has a RK-one element

G has no cut-points in any asymptotic cone