Joint work with K. Jankiewicz, A. Karrer, B. Sathaye.

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**Defn**: A CAT(0) group G is called boundary rigid if the visual boundaries of any two CAT(0) spaces on which G acts geometrically are homeomorphic.

## Bad news first:

**Thm:** (Croke-Kleiner 2000) There exists a CAT(0) group that is not boundary rigid.

$G = \mathrm{RAAG}(a - b - c - d)$.

## Good news:

- If $G$ CAT(0) and word hyperbolic, then $G$ is bdry rigid (Gromov?)
- If $G = \Gamma_1 \times \Gamma_2$ CAT(0) with $\Gamma_i$ hyp. then $G$ is bdry rigid (R)

- If $G = \Gamma \times \mathbb{Z}^n$ CAT(0), then $G$ is bdry rigid. (R)

- If $G = \Gamma_1 \times \Gamma_2$ with $\Gamma_i$ bdry rigid, then $G$ is bdry rigid. (Hosaka)

- If $G$ CAT(0) with IFP, then $G$ is bdry rigid. (Hruska-Kleiner)

<u>Favorite</u> <u>Question</u>: If $G$ a RACG is $G$ bdry rigid.

Remember... torsion restricts the actions!!!

Qing: You cannot build a C-K example using a RACG.

In the current work, we are considering groups acting geom. on the product of locally finite trees.

exs: $F_n \times F_m$ is boundary rigid by the above.

So we are really considering the irreducible kind - i.e. no finite index subgrp splits as a product.

These are known to exist by work of Burger-Mozes & Wise.
(simple CAT(0) groups & simple amalgams)

If $X = T_n \times T_m$, then $X$ is a CAT(0) space and can be given the structure of a CAT(0) cube (square) complex.
(each vertex has link $K(n,m)$ which is flag.)

**FACT:** The groups we consider all contain $\mathbb{Z}^2$ subgroups.

Thm: Sps $G$ torsion-free acts geometrically on product of trees and $G$ acts geom. on a CAT(0) cube cx space $X$, then $\partial X \cong \mathcal{C} * \mathcal{C}$ (join of Cantor sets).

Pf: If $X$ is CAT(0) cubed, then we can start off using a sledgehammer that we don't have in general.

- $G$ has no rank one elmt.

  since every geodesic line in $T_n \times T_m$ bounds a flat half plane.

- Rank Rigidity (Caprace/Sageev) implies $X \equiv X_1 \times X_2$ product of cube cx's.

- $\partial_\infty X \cong \partial_\infty X_1 * \partial_\infty X_2$

- $\partial_\infty X$ has dimension 1 (Geoghegan/Ontaneda) dim of $\partial$ is a QI-invariant.

- $\partial_\infty X_i$ has dim $= 0$.

You might think we are done but we aren't... because we do not have geometric group actions on $X_1, X_2$.

- Each $\partial_\infty X_i$ has at least 2 points and at least one has 3 or more points.

$\exists$ a $\mathbb{Z}^2 \leq G$ so $\exists$ flat plane $F$ in $X$, $\partial F \cong S^1$ in $\partial_\infty X$. So $|\partial_\infty X_i| \geq 2$. If they are both equal to 2, then $\partial_\infty X \cong S^1$! The only CAT(0) (non-hyp) groups with $S^1$ bdry are virtually $\mathbb{Z}^2$ which our group $G$ is <u>not</u>.

(Ruane)

- For $i=1,2$ $\partial_\infty X_i \cong \mathrm{Ends}(X_i)$

  $X_i$ is $\delta$-hyperbolic since it is "uniformly visible".

  Then $\partial X_i \xrightarrow{f} \mathrm{Ends}(X_i)$ is continuous and the fibers are connected components of $\partial_\infty X_i$.

  These are points since $\dim \partial X_i = 0$.

  $f$ a cont. bijection from a cmpt space to a Hausdorff space is a homeo.

- <u>Lemma</u>: For every cmpt $K \subseteq X$ and every open nbhd $U$ of an end (i.e. a point in $\partial_\infty X_i$) $\exists g$ s.t. $g \cdot K \subseteq U$.

This is true for the $G$-action on $X$ so is true for the $G$-action on $X_i$.

- The actual proof of Hopf's Ends Theorem applies to $\mathrm{Ends}(X_i)$ with the property from the previous lemma to show that if $|\mathrm{Ends}(X_i)| \geq 3$ then it is perfect. So $\partial_\infty X_i$ is perfect if $|\partial_\infty X_i| \geq 3$.
- One uses the topological characterization of Cantor set to conclude $\partial_\infty X_i \cong \mathcal{C}$.

Leftover: Can $|\partial_\infty X_1| = 2$ and $|\partial_\infty X_2| \geq 3$ ?

Answer: No, in this case $\partial_\infty X \cong \Sigma C$. but work of mine + Bridson implies $G$ is virtually $F \times \mathbb{Z}$ again, we know these groups are not.

Thus $\partial_\infty X \cong \partial_\infty X_1 * \partial_\infty X_2$

$\cong C * C$

General Case Requires some type of work around because we have 2 things to work around:

(1) No rank rigidity theorem in general

(2) Need to rule out that $X$ is a Euclidean building.

**Broad outline:**

Use a result of R. Ricks to show that the Tits bdry splits as a join.

This gets us around both points above and then we can follow the argument above to show $\partial_\infty X \cong C * C$.

Ricks result depends on work of Guralnik/Swenson where they outline a strategy for trying to prove rank rigidity in general.

This work requires a thorough understanding of the group action on the boundary.

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<u>Thm</u> (Ricks) $G \curvearrowright^{geom} X$ CAT(0) with $\dim \partial_T X = 1$. Then $\partial_T X$ splits as a join if $\exists$ closed, non-empty, proper $G$-inv. subset $\Lambda$ of $\partial X$ and a folded circle $K \subset \partial X$ s.t. $K \cap \Lambda$ consists of at most 2 antipodal pts. in $K$.

## Analogy:

In a hyperbolic group $G$, we know for $g \in G$ of inf order, $\bar{g} : \partial G \to \partial G$ is a homeo with 2 fixed pts - $g^{\pm\infty}$ - attracting and repelling fixed points.

A generic point of the boundary $p \in \partial G$ is a conical limit pt.
- i.e. $\exists \{g_n\} \subseteq G$ distinct and $a \neq b \in \partial G$

s.t. $g_n \cdot p \to a$ and
$g_n \cdot y \to b \quad \forall y \in \partial G - \{p\}$

- i.e. there is a point in the closure of the $G$-orbit of $p$ that looks like a repelling point of an attracting/repelling pair.

In Guralnik-Swenson, they generalize this notion to the CAT(0) setting where we no longer have a convergence group action. This requires the use of ultra-filters on $G$ - which is admittedly scary at first!!

However, here is a key observation to keep in mind:

For a fixed infinite order element $g \in G$, we have the following:

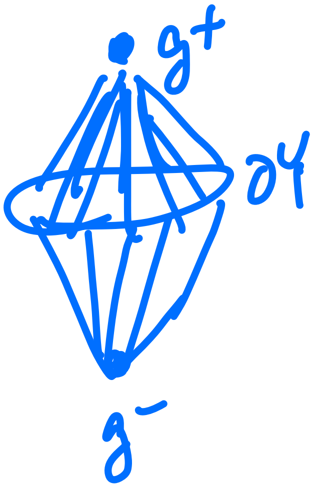
$g$ has an axis in $X - \ell_g$.

$\{\text{axes for } g\} = \mathrm{Min}(g) \cong Z \times \mathbb{R}$.

$\subseteq \{\text{parallels to } \ell_g\} \overset{=P(g)}{\cong} Y \times \mathbb{R}$.

$\bar{g} : \partial_\infty X \to \partial_\infty X$ as a homeo.

$\mathrm{Fix}(\bar{g}) = \partial P(g) = \sum \partial Y_i \ (R)$




This is an attracting set for the action of $\langle g \rangle$ on $\partial_\infty X$.

given $z \in \partial_\infty X \setminus \partial P(g)$.

$n \geq 0 \quad g^n \cdot z \to p \in P(g)$

you can determine where $z$ lands in this set if you know $\angle(z, g^-)$.

In particular, you can

give conditions that guarantee $p = g^{\infty}$.

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One interpretation of the Guralnik-Swenson work is that for a generic point $p \in \partial X$, some point in the closure of $G \cdot p$ looks like the "repelling point" of an infinite order element.