The computational complexity of knot genus in a fixed 3-manifold

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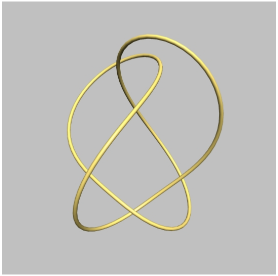
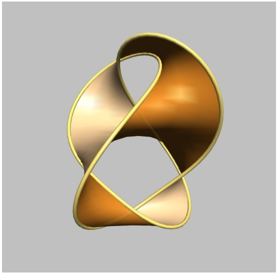
Unknot recognition, and knot genus

- Max Dehn posed the question of finding an algorithm for detecting the unknot. This led to the formulation of the *word and isomorphism problems*.
- The first algorithm for unknot recognition was given by Haken. Schubert defined the *genus* of a knot, and extended Haken's method to give an algorithm for determining the genus of a knot.





Unknot recognition, and knot genus

- Hass–Lagarias–Pippenger analysed the computational complexity of Haken's algorithm and gave an upper bound for its complexity.

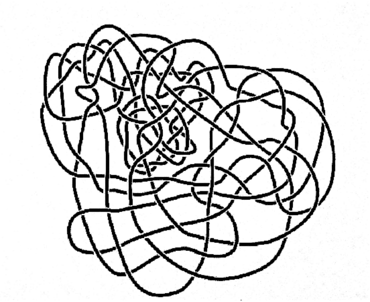


Figure: Haken's Gordian Unknot

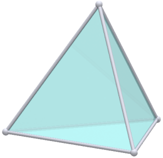
Computational complexity

- A problem lies in **NP** if it can be solved in polynomial time by a non-deterministic Turing machine.
- Intuitively, this means that whenever the answer is yes, the yes answer can be verified in polynomial time.
- Example of a problem that lies in **NP** for trivial reasons: COMPOSITE
- A nontrivial example of a **NP** question: PRIME (proved by Pratt)

Computational complexity

- A problem H is **NP**-hard if any problem L in the **NP** list can be reduced in polynomial time to H. Intuitively, H is as hard as the hardest problems in the **NP** list.
- A problem is called **NP-complete** if it lies in **NP**, and it is **NP**-hard.
- Examples of **NP-complete** problems: traveling salesman problem, subset sum problem, integer linear programming.

Input: A triangulated 3-manifold M, a knot $K \subset M$ given as a union of edges, and $n \in \mathbb{N}$.
Question: Is the genus of K less that n?
Size of the input: Sum of the number of tetrahedra and the number of digits of n.



Theorem (Agol–Hass–Thurston 2002)

The problem Upper bound for knot genus *is* NP-complete.

[Agol–Hass–Thurston] Is Determining knot genus in a fixed 3-manifold in **NP**?

Determining knot genus in a fixed 3-manifold

Input: A 3-manifold M given as surgery on a link $L \subset S^3$, a diagram of $K \cup L$, and an integer n.
Question: Is the genus of K equal to n.
Size of the input: Sum of the number of crossings of $K \cup L$ and the number of digits of n.

Theorem

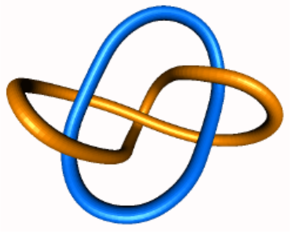
Let M be any fixed compact, orientable 3-manifold. The problem DETERMINING KNOT GENUS IN M *lies in* NP.

This was known for the case of $M = S^3$, or more generally rational homology 3-spheres, by the work of Lackenby.
Difficulty for the general case: For S^3 there is a unique homology class representing a surface that bounds the knot, but for general 3-manifold there can be infinitely many such homology classes.

A quick overview of the Thurston norm

- For any compact, orientable 3-manifold M, Thurston norm is a semi-norm defined on the second homology of M with $\mathbb{R}$ coefficients.
- It is a natural generalisation of the notion of knot genus.
- The norm of a homology class is obtained by minimizing the Euler characteristic (in absolute value) between its representative surfaces, after discarding any spheres and discs.
- The unit ball of the Thurston norm is a convex polyhedron with rational vertices.
- Thurston norm on $H_2(M, \partial M)$ defines a natural dual norm on the dual vector space $H^2(M, \partial M)$.
- The unit ball of the dual Thurston norm is a convex polyhedron with *integral* vertices.

Example: Whitehead link complement



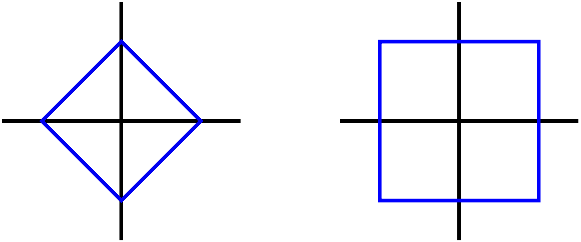


Figure: Left: unit ball of Thurston norm in H_2. Right: dual unit ball in H^2.

Ingredients of the proof of the main theorem

- THURSTON NORM OF A HOMOLOGY CLASS lies in **NP** (Lackenby's theorem). The proof of this theorem uses sutured manifold hierarchies and the associated taut foliations, and normal surface theory. In particular, our proof rests upon Lackenby's previous result for the case of rational homology spheres.
- Efficient certification of THURSTON NORM BALL FOR $b_1 \leq B$.
- An efficient simplicial basis for the homology of the knot complement.
- Controlling the number of faces of the Thurston norm in the presence of an efficient basis for the second homology.
- Non-compactness of the unit ball of the Thurston norm: We use normal surface theory to find an efficient basis for the subspace of homology with trivial Thurston norm, and to deal with reducible and boundary reducible 3-manifolds.
- The following question is in **NP** (Lackenby's theorem): Whether a compact orientable 3-manifold with toroidal boundary and $b_1 > 0$ is irreducible.

Efficient certification of Thurston norm ball for $b_1 \leq B$

Theorem

Fix an integer $B \geq 0$. The problem THURSTON NORM BALL FOR $b_1 \leq B$ *lies in* FNP*, where* b_1 *denotes the first Betti number.*

- **FNP** is the generalisation of **NP** from decision problems (where a yes/no answer is required) to function problems (where more complicated outputs might be required).
- **Warning**: This theorem uses a non-standard notion of complexity for its input. The size of the input is defined as sum of the number of tetrahedra in the triangulation plus the L^1 norm of a homology basis (rather than its number of digits). In particular, it can only handle 'small' inputs. It would be interesting to know if it lies in **FNP** with the more natural notion of complexity for the input.

Efficient basis for the (co)homology of knot complements

Theorem

Let M be a compact, orientable 3-manifold given by a surgery diagram, and K be a given knot in M with c crossings. There is an algorithm that builds a triangulation of the knot complement with $O(c)$ tetrahedra, together with an integral basis for the first cohomology with L^1 norm $O(c^2)$. All the implicit constants depend only on M and not the knot K.

Controlling the number of faces of the Thurston norm

Theorem

Let M be a compact, orientable 3-manifold. Assume that there are (properly immersed) surfaces $S_1, \cdots, S_b$ whose homology classes form a basis for $H_2(M, \partial M; \mathbb{R})$ and furthermore for each i we have $|\chi_-(S_i)| \leq m$. Then the number of facets of the Thurston norm ball is at most $(2m+1)^b$. Consequently, the number of vertices is at most $(2m+1)^{b^2}$ and the total number of faces is at most $b(2m+1)^{b^2}$.

Idea of the proof.

Facets of the unit ball of the Thurston norm correspond to the vertices of the unit ball of the dual Thurston norm on $H^2(M, \partial M; \mathbb{R})$. Thurston proved that the vertices of the dual ball are *integral* points. The inequality $|\chi_-(S_i)| \leq m$ gives a bound on the size of the coordinates of these integral points (in the dual space H^2) when written in a suitable basis. The number of integral points in a region of bounded size in $\mathbb{R}^b$ can be bounded from above.

Questions

Question

Does DETERMINING KNOT GENUS IN 3-MANIFOLDS WITH FIXED FIRST BETTI NUMBER *lie in* NP*?*

Question

Is there a sequence of compact orientable 3-manifolds M_i with triangulations $\mathcal{T}_i$ and with bounded first Betti number such that the number of vertices of the Thurston norm ball of M_i grows faster than any polynomial function of $|\mathcal{T}_i|$?