Equivariant Heegaard genus of reducible 3-manifolds

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based on joint work with Maggy Tomova

I. Heegaard splittings and group actions

Ingredients

1. Reducible 3-manifold
2. Heegaard splitting
3. Finite group of diffeomorphisms
4. Equivariant spheres and equivariant Heegaard surfaces

(*) Henceforth, only consider cpct. orientable 3-mflds where every sphere separates. Mostly closed, but if not no S^2 boundary components.

I. Heegaard splittings and group actions

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I. Heegaard splittings and group actions

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3. Finite group of diffeomorphisms

4. Equivariant spheres and equivariant Heegaard surfaces

I. Heegaard splittings and group actions

Definition: The equivariant Heegaard genus $\mathbf{g}(W;G)$ is the minimal genus of an equivariant Heegaard surface for W.

Question: Is $\mathbf{g}(W;\mathrm{G}) = \mathbf{g}(W\,|\,S;\mathrm{G})$?

I. Heegaard splittings and group actions

Theorem [T.]: There are examples s.t.

- $\mathbf{g}(W; \mathrm{G}) < \mathbf{g}(W|_S; \mathrm{G})$
- $\mathbf{g}(W; \mathrm{G}) = \mathbf{g}(W|_S; \mathrm{G})$
- $\mathbf{g}(W; \mathrm{G}) > \mathbf{g}(W|_S; \mathrm{G})$

I. Heegaard splittings and group actions

Theorem [T.]: If S is an equivariant system of reducing spheres for W splitting W into n irreducible 3-mflds, then

$\mathbf{g}(W;G) \leq \mathbf{g}(W|_S;G) + (c(|G|+1) - 2)(n - 1)$

$(c = 1,2)$

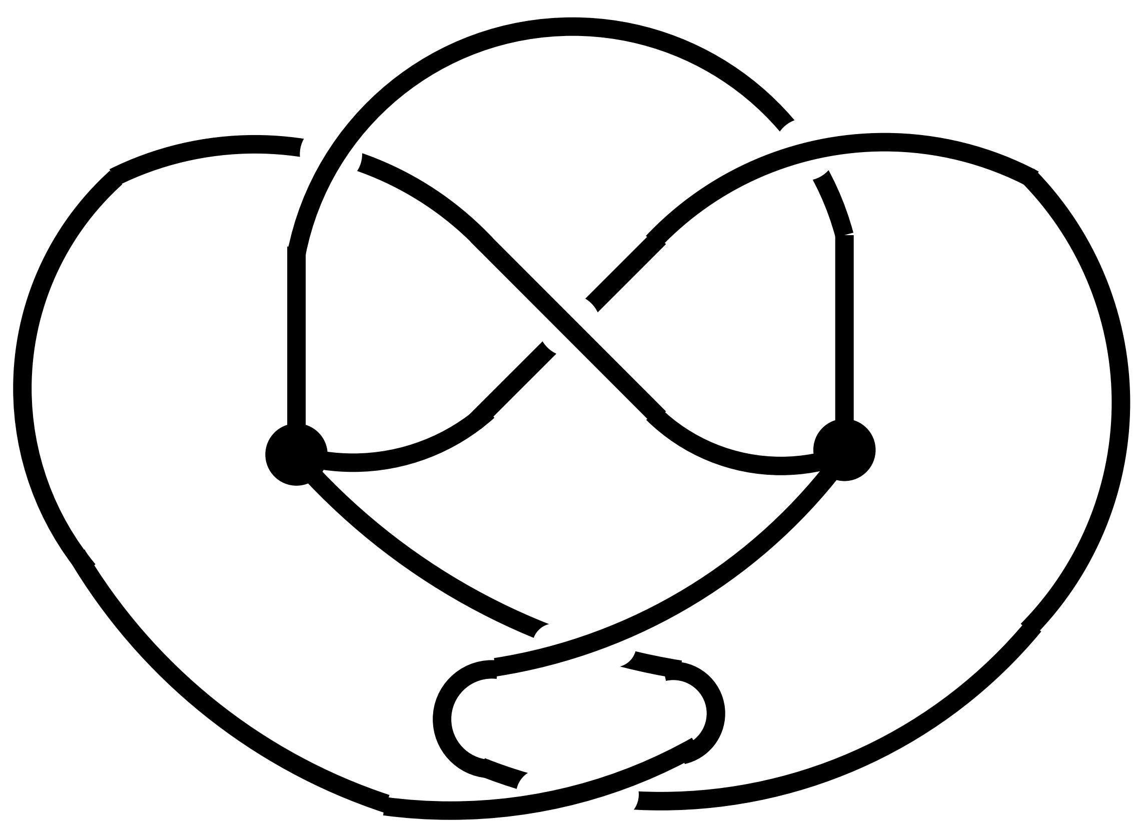
I. Heegaard splittings and group actions

Theorem [T.]: If S is an equivariant system of reducing spheres for W such that $W|_S$ has n orbits of components that are irreducible 3-mflds other than S^3 or lens spaces, then $\mathbf{g}(W;G) \geq 1 + n|G|/12$.

Theorem [T.]: If S is an equivariant system of reducing spheres for W such that each component of $W|_S$ is equivariantly comparatively small, then $\mathbf{g}(W;G) \geq \mathbf{g}(W|_S;G)$

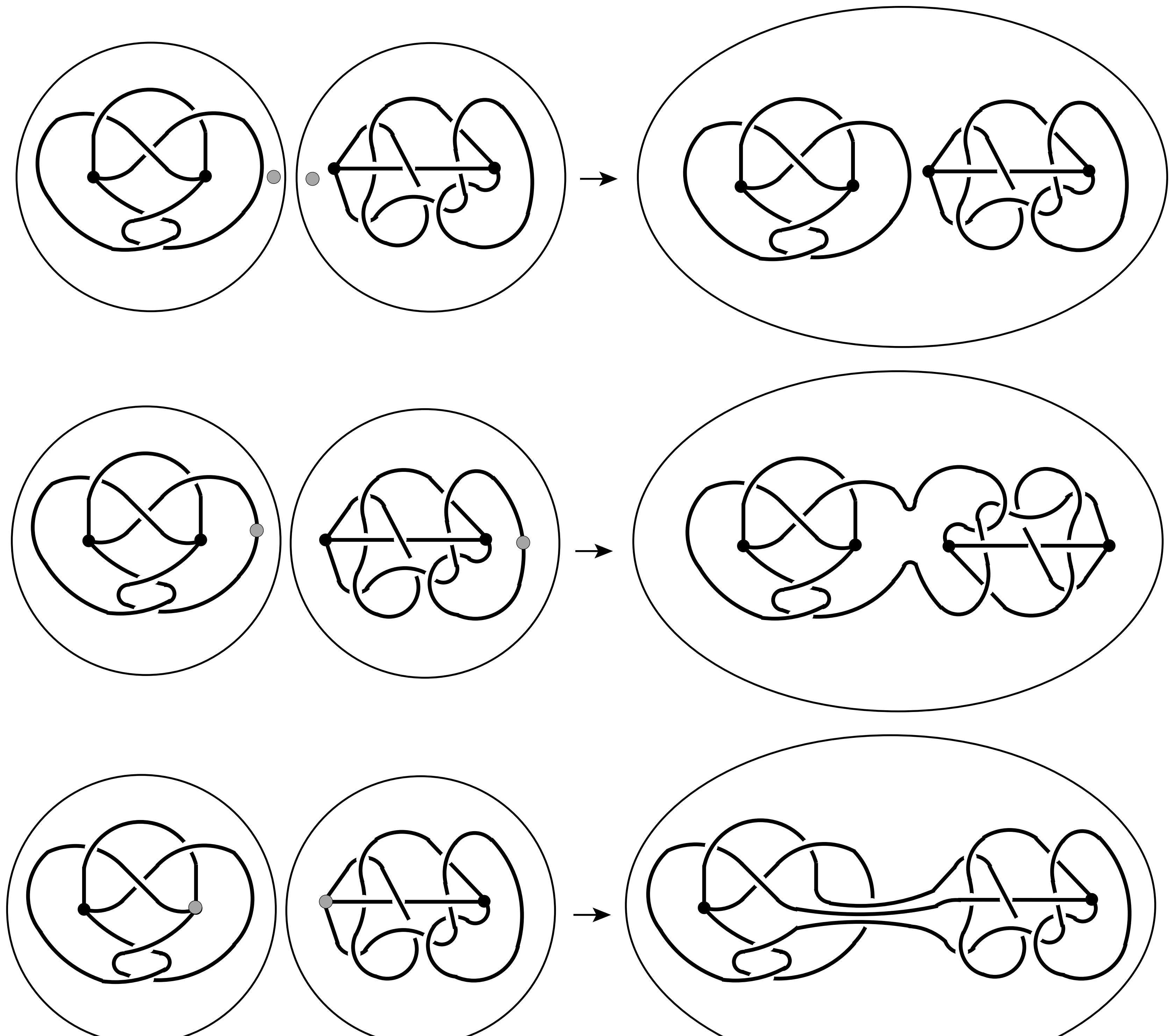
II. Orbifolds

Def: A 3-orbifold “is” a
(3-mfld, weighted trivalent spatial graph) pair.



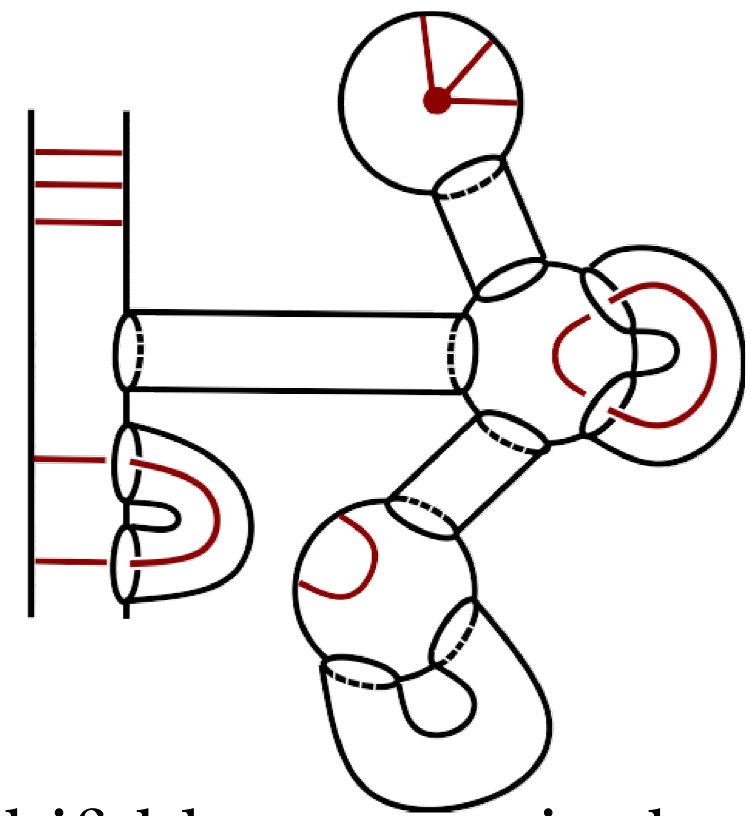
II. Orbifolds

Def: Orbifold sums



II. Orbifolds

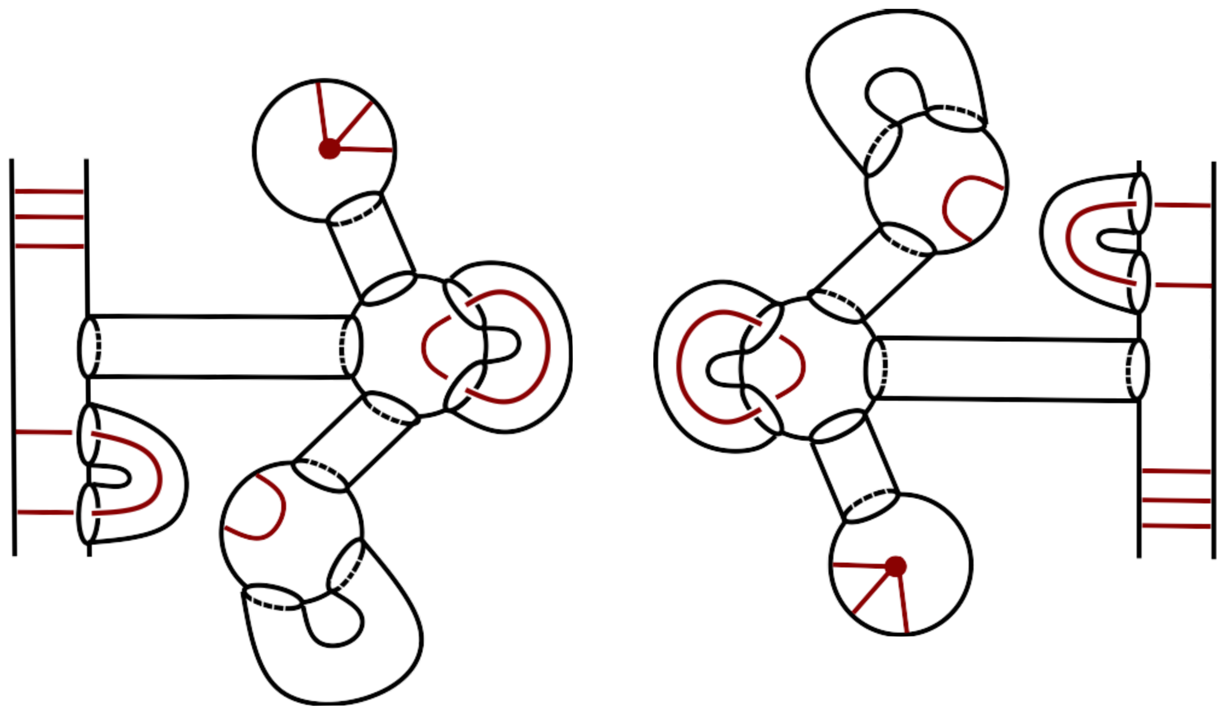
Def [Zimmermann]: Orbifold Heegaard splittings

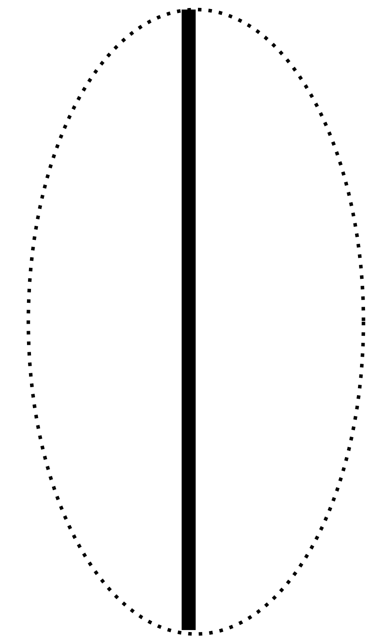


orbifold compressionbody

II. Orbifolds

Def [Zimmermann]: Orbifold Heegaard splittings

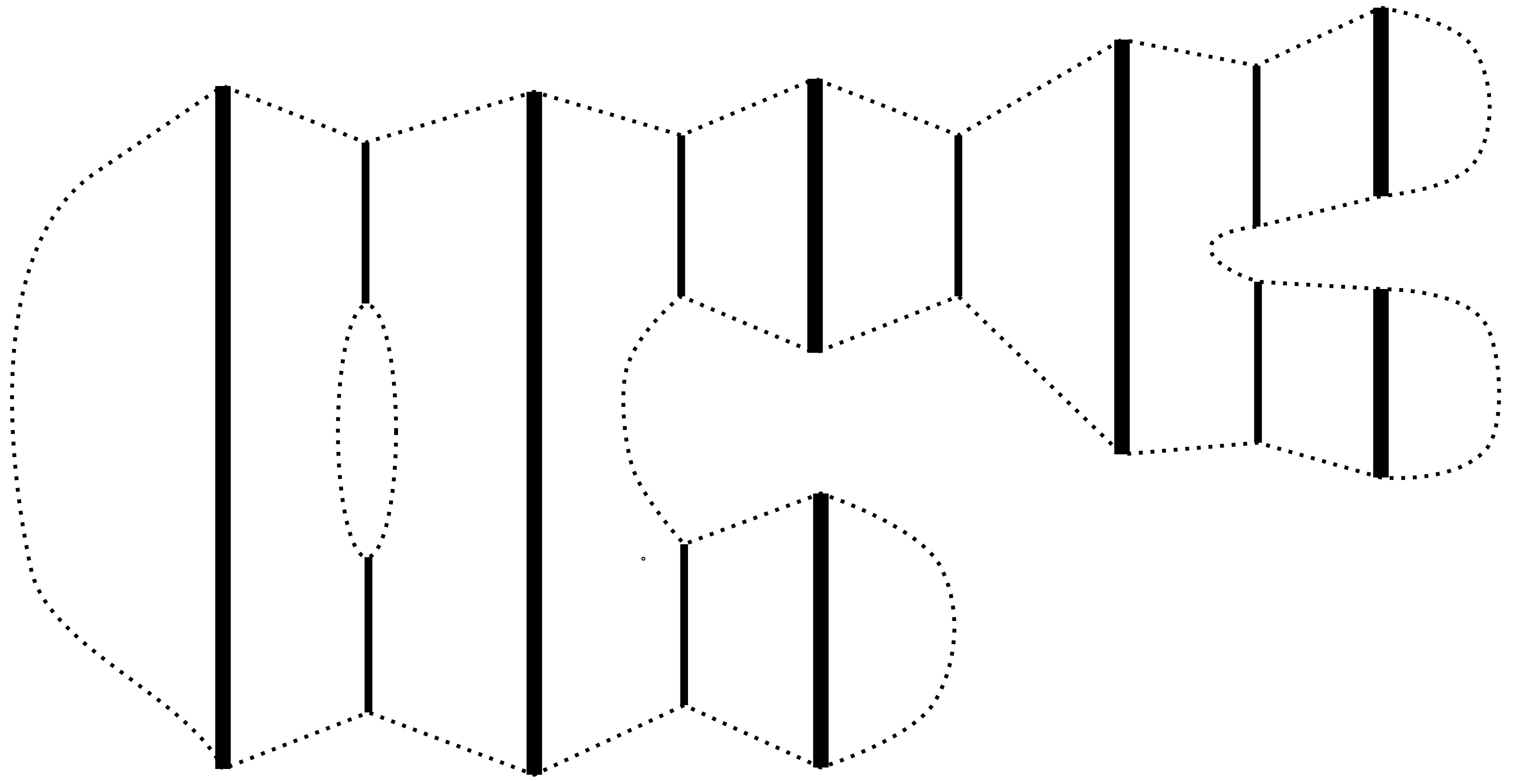




orbifold Heegaard surface

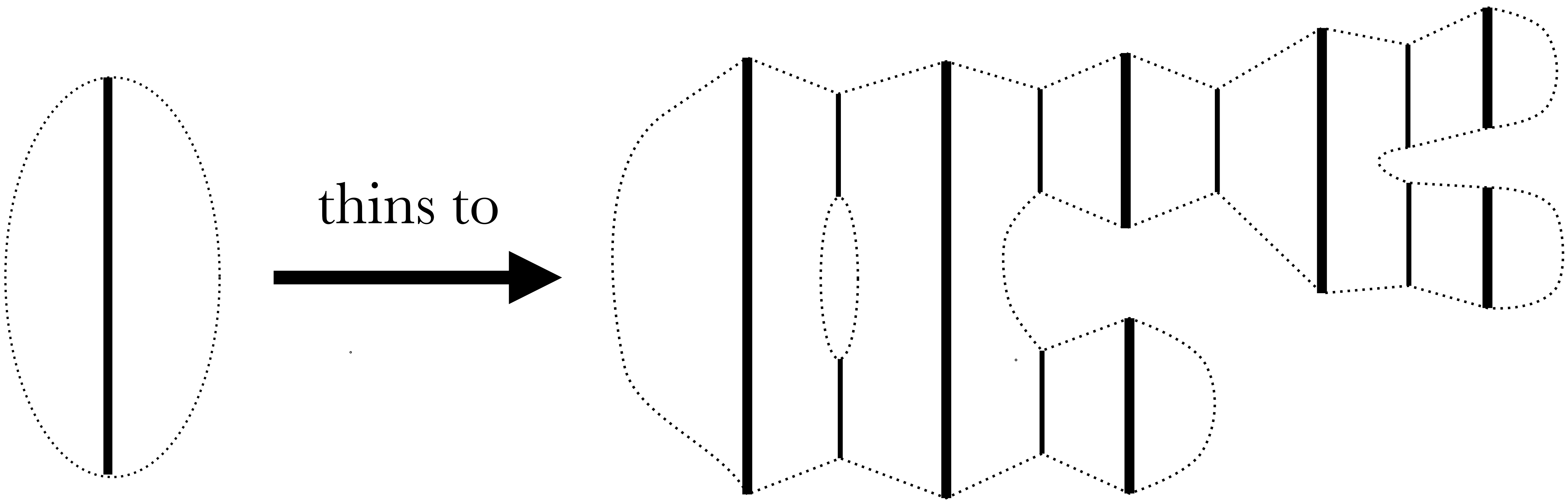
Def: Given a surface S ⊂ (M,T) its orbifold characteristic is:

II. Orbifolds



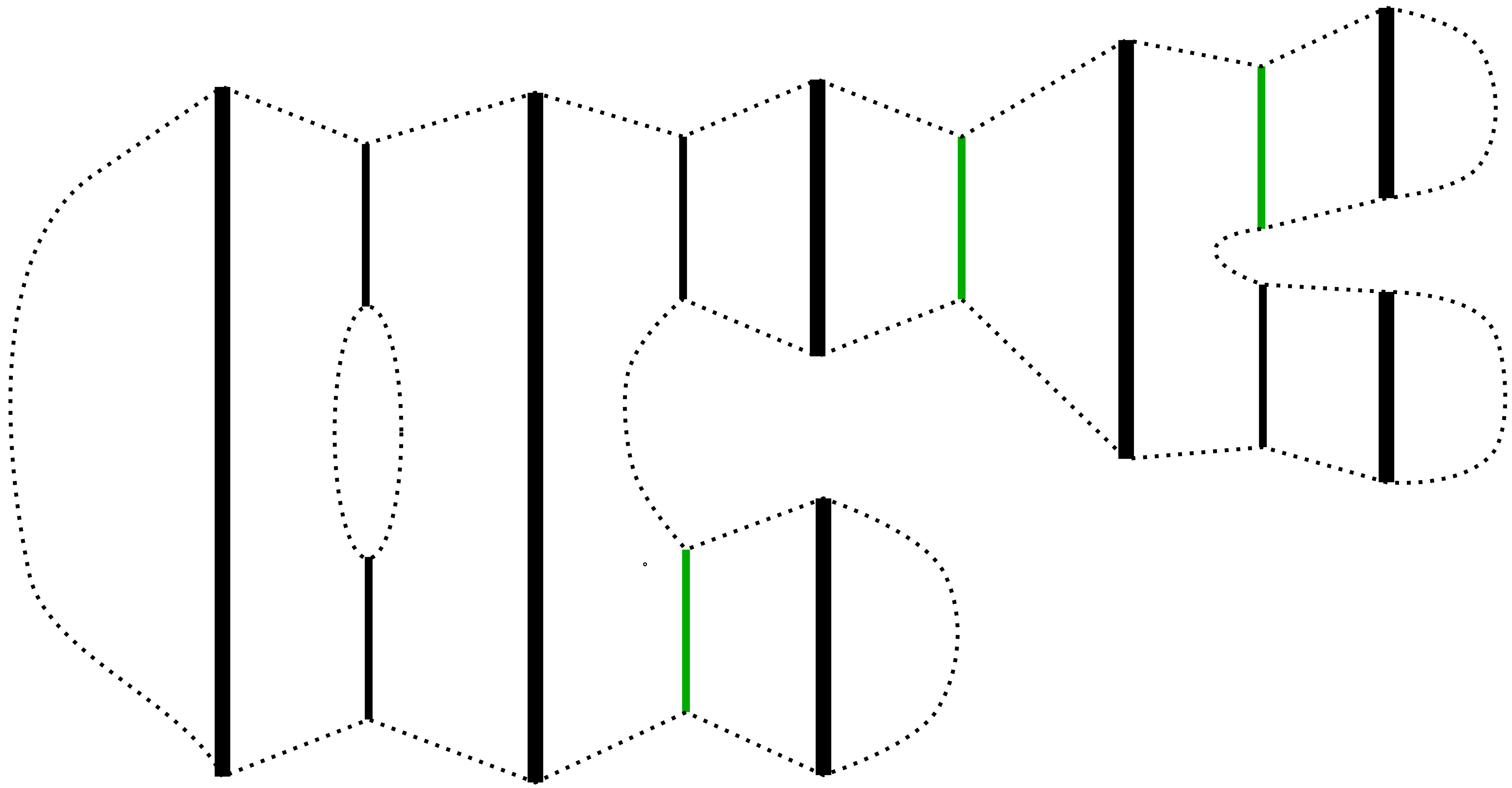
multiple orbifold Heegaard surface/
multiple vp-bridge surface

II. Orbifolds



Thm [T - Tomova]: Every multiple vp-bridge surface ***H*** can be thinned to a “locally thin” multiple vp-bridge surface.

II. Orbifold thin position and net orbifold characteristic

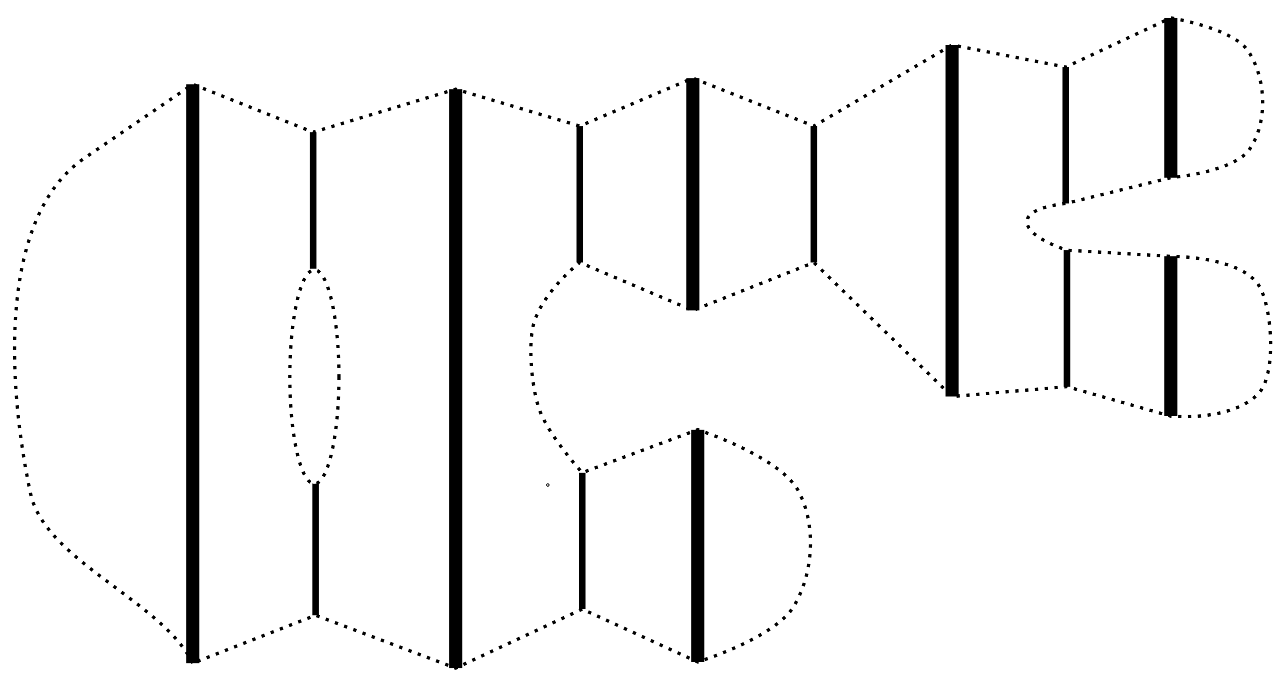


Thm [T - Tomova]: If ***H*** is locally thin, then the thin surfaces contain an efficient set of decomposing spheres.

II. Orbifold thin position and net orbifold characteristic

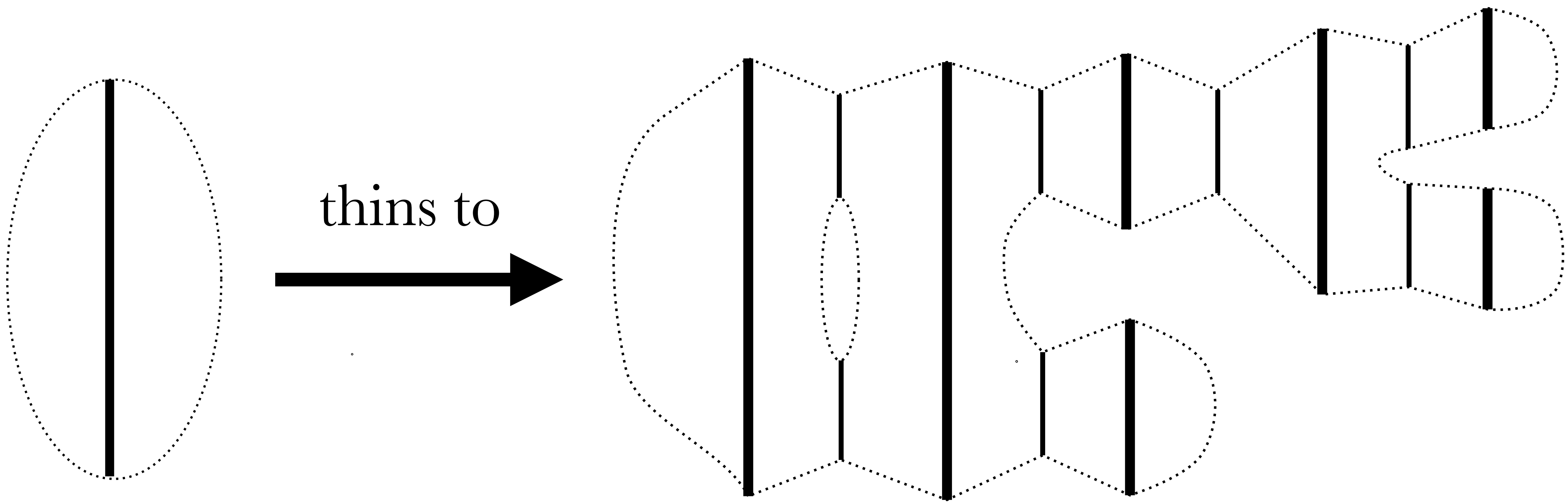
Def: Given a multiple vp-bridge surface $\boldsymbol{H} \subset$ (M,T) its net Heegaard characteristic is:

$$\text{net}\, x(\mathcal{H}) = x(\mathcal{H}^{+}) - x(\mathcal{H}^{-})$$



II. Orbifolds

Def: $\text{net}\, x(\mathcal{H}) = x(\mathcal{H}^+) - x(\mathcal{H}^-)$

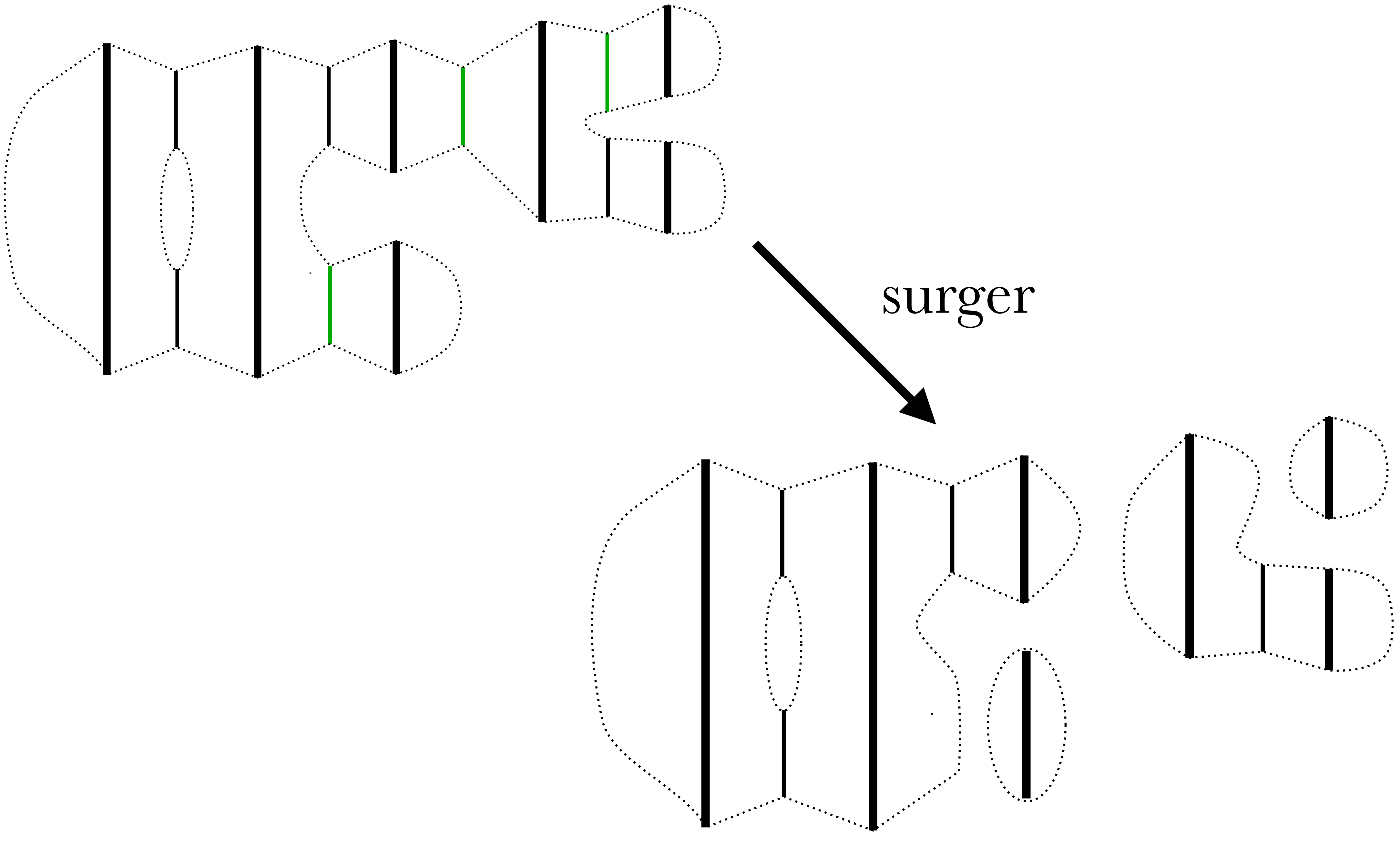


Thm [T.]: net Heegaard characteristic is non-increasing under thinning.

II. Orbifolds

Def: net $x(M,T)$ = min net $x(\boldsymbol{H})$

Thm [T.]: net $x(M,T)$ = net $x((M,T)\,|\,S)$ - $x(S)$



III. (Non)additivity

Thm [T]: There exists an equivariant system of spheres S for W such that net $x(W;G)$ = net $x(W|_S;G)$ - 2$|S|$
and $W|_S$ is irreducible.