Isotopy and Equivalence of Knots in 3-manifolds

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A knot is a (tame) embedding $K : S^1 \hookrightarrow M^3$.

Two knots $K, J$ in $M$ are

1. Equivalent: If $\exists$ o.p. homeo $h : M \rightarrow M$ s.t. $h(K) = J$

2. Isotopic: If $\exists$ 1-parameter family of embeddings $f_t : S^1 \times [0,1] \rightarrow M$ s.t. $f_0 = K$ and $f_1 = J$

3. Ambient Isotopic: If $\exists$ 1-parameter family of homeos $h_t : M \times [0,1] \rightarrow M$ s.t. $h_0 = id$ and $h_1(K) = J$.

$\Rightarrow \Rightarrow \Rightarrow$ 1, 2

2 $\Rightarrow$ 3 (isotopy extension) 1 $\Rightarrow$ 3 $\Rightarrow$

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Thm (Fisher, ’66) $\text{Homeo}^+(S^3)$ is path-connected

1, 2, 3 equivalent for $S^3$

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Def: $\text{Mod}^+(M) = \pi_0(\text{Homeo}^+(M)) = \frac{\text{Homeo}^+(M)}{\text{isotopy}}$

Mapping class group of $M$
Q: Does equivalence imply isotopy in general $M$?

A: No!

\[ \{ \text{Free homotopy classes of loops} \} \leftrightarrow \{ \text{Conjugacy classes in } \pi_1(M) \} \]

$\implies$ If $h : M \to M$ acts nontrivially on conjugacy classes (e.g. on $H_1$), then $K$ and $h(K)$ may not be ambient isotopic.

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**Thm. 1 (ABDPR)** Suppose $M$ is a prime, orientable, closed 3-mfd and the $\text{Homeo}^+(M)$ fixes every isotopy class of knot in $M$. Then $h$ is isotopic to the identity.

$\text{Homeo}^+(M)$ "sees" knot isotopy classes.

**Corollary:** $M$ prime, orientable, closed.

Then isotopy $\iff$ Equivalence iff $\text{Mod}^+(M) = \mathbb{Z}$. 

M prime $\implies$ $M$ irreducible, "every isotopy" $\leftrightarrow$ "every homotopy" 

$M = S^1 \times S^2$, need "every isotopy"
$M = S^1 \times S^2$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\begin{align*}
1 \rightarrow \mathbb{Z}_2 & \rightarrow \text{Mod}^+(S^1 \times S^2) \rightarrow \mathbb{Z}_2 \rightarrow 1 \\
\text{Gluck twist} & \uparrow \gamma
\end{align*}$

$\Rightarrow -1 \in \pi_1(S^1 \times S^2) = \mathbb{Z}$.

Gluck twist $\pi_1(S^0(3)) = \mathbb{Z}_2$

**Thm 2 (ABDPR)** For every winding number $w \neq 0 \in \mathbb{Z}$, there is $K$ with winding number $w$ s.t. $g(K) \neq K$. If $w$ is odd, $\gamma(K) \sim K \Rightarrow K \cong S^1 \times \mathbb{R}$

$\Rightarrow$ Thm 1 when $M = S^1 \times S^2$

**M irreducible**

$\text{Mod}^+(M) \hookrightarrow \text{Out}(\pi_1(M)) = \frac{\text{Aut}(\pi_1)}{1\text{im}}$ (Many people...)

We show if $h: M \to M$ acts trivially on conjugacy classes then $h^* = 1 \in \text{Out}(\pi_1) \Rightarrow h \sim 1$.

**Def (Grossman ’75)** A group $G$ has **Property A** if every conjugacy class preserving automorphism is inner.

E.g. Abelian groups have Prop. A.

**Thm 3 (ABDPR)** Every orientable 3-mfd group has Property A.

$\Rightarrow$ Thm 1 when $M$ irreducible.
**Idea of Proof**: Use the Prime and JSJ-decomposition

1. Free products (Neshchadim ’96)
   \[\Rightarrow\text{Non-prime 3-manifolds have Prop.} A\]

2. Hyperbolic + relatively hyperbolic (Minasyan-Osin ’10)
   \[\Rightarrow\text{Hyperbolic JSJ-components have Prop.} A.\]

3. Most Seifert fibered 3-mfd groups with or without bdy
   (Allenby-Kim-Tang, ’03, ’09) All except \(S^2(p,q,r)\) and \(T^2(p)\).
   We finished remaining cases.

- If \(M\) non-prime or has trivial JSJ-decomposition \(\checkmark\)
- Else: \(M\) is Haken.
  Waldhausen \(\Rightarrow\) Any \(\phi \in \text{Out}(\pi_1)\) rep by homeo \(h: M \to M\)
  unique up to isotopy.

\[h\text{ class-preserving}\]
\[h\text{ preserves JSJ-decomposition}\]
\[\text{Restriction to each piece class-preserving}\]
Grossman (’75) introduced Property A and showed:

- Prop A + Conjugacy separable $\Rightarrow$ Out $(G)$ is residually finite.

- Surface groups, free groups have Prop. A.
  $\Rightarrow$ Mod $(S_g)$, Out $(F_n)$ are residually finite.
  unknown $\uparrow$ not linear, $n \geq 3$

Q: Are all 3-mfd groups conjugacy separable?